GAUS-AG: LEARNING SEMINAR ON DIAMONDS, WITH AN INCLINATION TOWARDS TOPOS THEORY

DMYTRO RUDENKO*, RONALD SOLODOV†

1. Introduction

The main motivation for the develpment of the theory of diamonds (from [Sch17]) was to find a theory of spaces in which moduli spaces of mixed characteristic shtukas could live, and where their étale cohomology could be computed, realizing the local Langlands correspondence. As explained in [SW], one problem was to define something like $\operatorname{Spa}\mathbb{Q}_p \underset{\mathbb{F}_1}{\times} \operatorname{Spa}\mathbb{Q}_p$, over which the moduli space of mixed characteristic shtukas should be fibered. The problem is that $\operatorname{Spec}\mathbb{Z} \times \operatorname{Spec}\mathbb{Z} = \operatorname{Spec}\mathbb{Z}$ if the coherent information (rings) is taken into consideration while forming the fibre product. If one were to forget this information and retain only some necessary topological information, then the product would make sense in some topological sense (like product of two topological spaces). To every analytic adic space X over $\operatorname{Spa}\mathbb{Z}_p$ one can associate the correponding diamond $X^\diamondsuit$, and this should be thought of exactly as of extracting this necessary topological information from X, while forgetting the structure morphism to $\operatorname{Spa}\mathbb{Z}_p$. Since the étale site is "topological" (it is invariant under universal homeomorphisms), by passing to $X^\diamondsuit$ one does not lose the étale information, in fact, one can compute étale cohomology of any diamond. By now we should all agree, looking at [FS], that the motivation behind the theory of diamonds has been justified and the theory has proven to be successful.

While the primary aim of the paper [Sch17] is to develop a six functor formalism for étale cohomology general enough to be applied to p-adic stacks (as opposed to just p-adic spaces such as adic or Berkovich, but in a way analogous to Artin's algebraic spaces) and thereby used in the geometrization of the local Langlands correspondence as in [FS], the paper has proven to contain numerous propositions and techniques that are worthy in their own right. For instance, the paper [Sch17] provides yet another example of a six functor formalizm and its construction, in fact, the first six functor fomralizm that Lucas Mann constructed by following the general construction recipe as in [M] or [HM] was the six functor formalizm on small v-stacks (that include diamonds). We also believe that it is interesting to understand how far the analogy between diamonds and algebraic spaces goes, because diamonds after

*rudenko@math.uni-hannover.de
†solodov@math.uni-hannover.de

all have topological nature, as opposed to algebraic spaces, which also capture coherent information. Arguably, this should be studied before one goes on to genuine analytic stacks.

In this learning seminar we will try to study a part of the technical backbone of [Sch17] as something which is generally good to know, while leaving the construction of the six functor formalism out. We believe that this will enable the participants to learn this – and many other theories – when the time comes. We also believe that the pervading topological methods in [Sch17] such as the theory of spectral spaces and their points, various factorizations of maps of spaces, and sheaves/stacks together with cohomology, allow us to elicit a general theme for our discussion: we will be topologically (more precisely, topos-theoretically) inclined.

Remark. In talk descriptions we tried to avoid the style "discuss theorem xyz, prove lemma 123, state conjecture abc" in order to give more freedom to the speakers and make it look less dry, given that the paper [Sch17] itself is already very technical. If a speaker is not sure what to include in their talk, they are very welcome to contact the organizers! In a perfect world, this can be also adjusted according to the preferences of all participants.

Remark. In theory, one could give the definition of a diamond arbitrarily early in the seminar. However, we believe that getting used to the underlying topological theory (which we deem appealing in its own right) beforehand will help to understand diamonds; also, we did want to include and learn some technical stuff. On the other hand, of course, there is room even in the first talks to insert first examples of diamonds that occur from adic spaces. We explicitly ask the speakers to include examples in their talks.

Remark. In conjunction with the previous two remarks, we understand that the paper [Sch17] is difficult to read in the order in which it is written. We are trying to make a statement that slightly infusing the first sections of [Sch17] with some outside context will make them more palatable. At the same time, we are quite sure that the participants will look further in the paper faster than we progress, which we find natural: after all, this is a learning seminar.

Prerequisites: the participants are assumed to be familiar with the content of [Sch13, Sections 3 and 4], as well as to have a basic idea about stacks, for example, as in [Vis].

Time: Every second Friday there will be two talks, each not more than 90 minutes with questions, from 2pm till 6pm, starting on the 23rd of October.
Place: By default it is Leibniz University Hannover, Room A410 and via Zoom (Zoom link will be communicated).

2. Distribution of the talks

Talk 0: Distribution of the talks. (Oct 23, Speaker: Dmytro)
The speaker will give a very brief summary of the talks, and agrue why topos theory is relevant. We will then distribute the talks.

Talk 1: Spectral spaces and various Stone dualities. (Oct 23, Speaker: Dmytro)
This talk should cover the most important statements from [Sch17, Section 2]. Also, it will be used several times that the category of spectral spaces is dual to the category of distributive lattices. At the same time, it is equivalent to the pro-category of finite posets. This is parallel to the story with profinite spaces, which will of course play an equally important role, and therefore should be explained. **Bonus:** Finite posets are dual to finite distributive lattices. Definition of a frame and the dual concept of locale. The categorification of a locale is what is called a Grothendieck topos, which is especially clear with the definition via Giraud's axioms. Spectral spaces are like coherent topoi.

Talk 2: Perfectoid spaces and tilting equivalence. (Oct 23, Speaker: Tian)
After this talk we should end the day by recalling perfectoid spaces and the tilting equivalence, which are the most important points of [Sch17, Section 3]. The speaker should make an accent on the characterization of untilts of a perfectoid ring of characteristic p, as in [Sch17, Theorem 3.17]. Omit the discussion of almost mathematics.

Talk 3: Points of analytic adic spaces and monomorphisms of representable presheaves on Perfd**.** (Nov 6, Speaker:)
The speaker should follow [Sch17, Section 5]. They should necessarily give a thorough account of vertical specialization/generalizations of points in analytic/perfectoid adic spaces, see, e.g., [W]. In particular, it should be explained that for $x \in X$ the adic spectrum $\mathrm{Spa}(K(x), K(x)^+)$ is totally ordered by specializations possessing one closed and one generic point. Discuss [Sch17, Proposition 5.3] and the valuative criterion of separatedness [Sch17, Proposition 5.11] in detail (with proofs), which, if done correctly, should be almost identical to what is done in schemes.

Talk 4: Proposition 6.4. (Nov 6, Speaker:)
The speaker should explain in detail [Sch17, Proposition 6.4] and its proof.

Talk 5+6: (Strictly) totally disconnected perfectoid spaces. (Speakers:)
In this talk we will apply and dive deeper into the theory of spectral spaces. The speakers

should follow [Sch17, Section 7]. [Sch17, Lemma 7.6] can be omitted. Mention local topological spaces and explain the analogy between these and w-locality. Explain that for us strictly totally disconnected perfectoid spaces play role of w-local schemes from the paper [BS, Section 2] (what about the directly analogous notion of w-local perfectoid spaces?). The speakers should make an accent on the universal strictly totally disconnected perfectoid pro-étale cover (see [Sch17, Lemma 7.18]), and if possible, attach it to the discussion of the universal pro-finite étale cover (which is by the way the universal π_1-torsor in a rigorous topos-theoretic sense). **Bonus:** Connected components of w-local schemes are extremally disconnected, which allows for a more elaborate description of the map of sites from the pro-étale site of a scheme to the pro-étale site of a point. What to do in our case of strictly totally disconnected perfectoids, is there an analogous map of sites? See the discussion around [MW, Definition 1.1] and [MW, Lemma 1.2].

Talk 7: The pro-étale and v-sites. (Speaker:)
This is section 8. Explain the difference between the pro-étale site defined there and the one defined in [Sch13], for example, while comparing [Sch17, Proposition 8.5] to [Sch13, Corollary 3.17]. Also, draw the very enlightening analogy between the v-site and the usual v-site for schemes. This section 8 starts with some generalities on topos theory, which should be reinforced with the following. Explain the relation between algebraic topoi and coherent topoi. Draw analogy between coherent topoi and spectral spaces if was not done in talk 1. **Bonus:** Since we will be working with stacks rather than sheaves at some point, it will be interesting to say here something about $(2,1)$-topoi as opposed to $(1,1)$-topoi. In the context of section 8 it would be reasonable to explain the content around [DAG, Definition 3.1].

Optional Talk 8: Stacks on the pro-étale and v-sites. (Speaker:)
Cover section 9. Skip [Sch17, Lemma 9.4] and [Sch17, Lemma 9.5]. This talk is more or less flexible and one can digress on the theory of stacks in groupoids (giving rise to $(2,1)$-topoi), or even better on the theory of stack in categories (giving rise to $(2,2)$-topoi), see [Vis] for this. **Bonus:** Tell us exactly how the theory of $(2,1)$-topoi is subsumed by the theory of $(\infty,1)$-topoi, and argue that it does not subsume the theory of $(2,2)$-topoi.

Talk 9: Morphisms of v-stacks. (Speaker:)
Do section 10. **Bonus:** In the context of [Sch17, Def. 10.7] it would be reasonable to discuss the content around [HTT, Definition 5.5.6.1] and especially [HTT, Lemma 5.5.6.15]. Do not omit [Sch17, Proposition 10.9]. The example at the end with torsors is probably nice.

Talk 10: Diamonds. (Speaker:)
Follow [Sch17, Section 11]. It is a very big section and the speaker is free to decide (in the course of the seminar together with the other participants) what to state without proof and what not to mention. [Sch17, Proposition 11.9] is important. [Sch17, Example 11.12] is probably interesting to everyone. [Sch17, Proposition 11.13] is important. [Sch17, Proposition 11.23] can be stated as well. The concept of being locally spatial (diamond) is also important and useful in practice.

Talk 11: Analytic adic spaces as diamonds and other examples. (Speaker: Simon)
This talk is a continuation of the previous talk and is about examples of diamonds. The most relevant is section 15. The goal of this talk is to explain that every analytic adic space over $\operatorname{Spa}\mathbb{Z}_p$ has the associated diamond whose étale site coincide with that of the adic space. After that, give examples of diamonds that apper often and come from adic space, and then those that do not come from adic spaces.

Talk 12: Small v-stacks. (Speaker:)
This is about [Sch17, Section 12], which generalizes [Sch17, Section 11] from diamonds to small v-sheaves and small v-stacks. Technically, many proofs are the same as in section 11. [Sch17, Theorem 12.18] is important here.

Optional Talk 13: Berkovich spaces enter the game. (Speaker:)
The section 13 contains further foundational technicalities, perhaps the most interesting part of it being the role of Berkovich spaces there. One can concentrate on [Sch17, Definition 13.7] and [Sch17, Proposition 13.12]. The latter proposition is some interesting factorization result about spectral/compact Hausdorff spaces. **Bonus:** The material about Berkovich spaces may be accompanied by the discussion after [HTT, Proposition 7.3.6.9] about the relation to topoi.

Talk 14: Comparison of étale, pro-étale, and v-cohomology. (Speaker:)
This is about the comparison of corresponding topoi. The speaker should explain [Sch17, Lemma 14.5] in detail, and may decide whether or not to omit the technicalities on completions of derived categories around [Sch17, Proposition 14.10], but [Sch17, Theorem 14.12] is important. **Bonus:** Whether the derived category is complete depends on the topos in question, which can be e.g. *replete*, see [BS, Section 3]. Relate it to the notion of a *Postnikov complete* $(\infty,1)$-topos, see [MR] specifically and see [H, Section 1] for a nice background on hypercompletness and Postnikov completness.

Optional Talk 15: The six functor formalism. (Speaker:)
State the six functor formalism for the étale cohomology of diamonds (or even small v-stacks) constructed in [Sch17]. **Bonus:** State it in the language of [HM].

References

[BS] B. Bhatt and P. Scholze, *The pro-étale topology for schemes*, arXiv:1309.1198, 2014.

[DAG] J. Lurie, *Derived algebraic geometry VII: Spectral schemes*, link, 2011.

[FS] L. Fargues, P. Scholze, *Geometrization of the local Langlands correspondence*, arXiv:2102.13459, 2021.

[H] P. J. Haine, *Descent for sheaves on compact Hausdorff spaces*, arXiv:2210.00186, 2022.

[HM] C. Heyer, L. Mann, *6-Functor Formalisms and Smooth Representations*, arXiv:2410.13038, 2024.

[HTT] J. Lurie, *Higher Topos Theory*, Princeton University Press, 2009.

[M] L. Mann, *A p-Adic 6-Functor Formalizm in Rigid Analytic Geometry*, arXiv:2206.02022, 2022.

[MW] L. Mann, A. Werner, *Local systems on diamonds and p-adic vector bundles*, arXiv:2005.06855, 2020.

[MR] S. Mondal, E. Reinecke, *On Postnikov completeness for replete topoi*, arXiv:2210.14146, 2022.

[MW] L. Mann, A. Werner, *Local systems on diamonds and p-adic vector bundles*, arXiv:2005.06855, 2020.

[St] M. Stone, *Topological representations of distributive lattices and Brouwerian logics*, Časopis pro pěstování matematiky a fysiky, Vol. 67 (1938), No. 1, 1–25.

[Sch13] P. Scholze, *p-adic Hodge theory for rigid-analytic varieties*, Forum of Mathematics, Pi. 2013.

[Sch17] P. Scholze, *Étale cohomology of diamonds*, arXiv:1709.07343, 2017.

[SW] P. Scholze, J. Weinstein, *Berkeley lectures on p-adic geometry*, link, 2017.

[Vis] A. Vistoli, *Notes on Grothendieck topologies, fibered categories and descent theory*, arXiv:math/0412512, 2004.

[W] T. Wedhorn, *Adic spaces*, arXiv:1910.05934, 2019.