SEMINAR ON PERVERSE SHEAVES AND INTERSECTION COHOMOLOGY

This seminar introduces several tools useful in the study of singular spaces and varieties. We begin by studying constructible sheaves on stratified spaces and the perverse t-structure on them. While we follow a modern formulation in the language of ∞-categories, in particular [Hai24; HPT26], there is a lot of classical literature on this subject [BBD82; GM80; KS90; Dim04]. Next, we introduce intersection cohomology, a cohomology theory for singular spaces which enjoys a form of Poincaré duality, and study some of its main properties. It was originally constructed by Goersky-MacPherson using singular simplices satisfying a certain transversality condition, but we will follow a sheaf theoretic approach due to Deligne. In the second half of the seminar, we focus on the Beilinson-Bernstein-Deligne-Gabber decomposition theorem as the main application of this theory, following the Hodge-theoretic proof from [CM05], and study some applications of this result.

Talk 1: (20.10.26) Introduction (Dominik)

Talk 2: (27.10.26) Stratified spaces and constructible sheaves
Introduce the notion of a stratified space over a poset and morphisms between them [Hai24, Section 1.2], [AFR19, Section 1.1]. Quickly recall the six functor formalism of sheaves on (nice) topological spaces. Introduce the category of constructible sheaves on a stratified space and explain its compatibility with $*$- and !-functors [HPT26, Section 2.2 and 2.5]. Mention that later we will mostly be interested in those with perfect stalks [HPT26, Notation 4.1.1]. Introduce the exit path category and state [Lur17, Theorem A.3.2] [PT26, Theorem 2]. More classical literature on this subject can for example be found in [Dim04, Section 4].

Talk 3: (03.11.26) Perverse sheaves (Joao)
Recall the notion of a t-structure on a stable category and introduce the standard t-structure on (hypercomplete) sheaves on a topological space as the main example for us. Prove how to glue two t-structures in a recollement (e.g. [Ban07, Theorem 7.2.2]). Define the perverse t-structure on the category of hypersheaves on a stratified topological space and prove that it is indeed a t-structure [HPT26, Section 3]. Explain how it restricts to a t-structure on constructible sheaves [HPT26, Remark 4.2.12]. More classical literature can be found in [Dim04, Section 5.1] or [Ban07, Section 7].

Talk 4: (10.11.26) Intersection cohomology 1: Definitions and basic examples (Jordan)
Introduce topological pseudomanifolds and give examples [Ban07, Section 4.1.2]. Introduce the intermediate extension functor $j_{!*}$ and prove its alternative characterisation by iterated truncations [Dim04, Definition 5.2.6, Proposition 5.2.10], [BBD82, pp. 1.4.22, 1.4.24, 2.1.11]. Define intersection cohomology with respect to a local system L by $IC(L) := j_{!*}L[\dim X]$ and

$IH^k(L) := \mathbb{H}^{k-\dim X}(X, IC(L))$, see [Ban07, Eq. 4.13] or [Dim04, Definition 5.4.3]. Compute intersection cohomology in the smooth case and in the case of isolated singularities [Dim04, Proposition 5.4.4].

Talk 5: (17.11.26) Intersection cohomology 2: Independence from the stratification (Timo)
Introduce the axioms [AX] from [Ban07, Definition 4.1.13, Remark 4.1.14 (i)] and prove that intersection cohomology is the unique sheaf satisfying these axioms [Ban07, Theorem 4.2.1]. Prove independence of intersection cohomology from the stratification [Ban07, Theorem 4.3.1].

Talk 6: (24.11.26) Intersection cohomology 3: Poincaré duality (Klaus)
Prove the support and cosupport conditions for perverse sheaves [Dim04, Proposition 5.1.16], [KS90, Proposition 10.2.4 and 10.2.5]. Recall Verdier duality for the six functor formalism of sheaves on topological spaces and the duality functor $\mathbb{D}_X = \hom(-, \omega_X)$. Explain how duality interchanges j_* and $j_!$. Use the support and cosupport conditions to show how $\mathbb{D}_X$ interchanges the p-connective and q-coconnective part of the perverse t-structure for complementary perversities p and $q(n) = -n - p(n)$. From this, prove that $\mathbb{D}_X IC_p(L) \simeq IC_q(L^\vee)$, where $L^\vee$ is the dual local system of L and deduce Poincaré duality for intersection cohomology. Also see [Ban07, Theorem 4.4.1].

Talk 7: (01.12.26) Examples and computations (Luca)
Compute intersection cohomology for the cone over a manifold with boundary (see [Ban07, Example 4.1.12] for the result, but use an argument similar to the proof of [Dim04, Proposition 5.4.4] for the computation). Prove the Lefschetz hyperplane section theorem for intersection cohomology [Dim04, Theorem 5.4.6]. Discuss the example of an affine cone [Dim04, Example 5.4.15 (ii)].

Talk 8: (08.12.26) Statement and some consequences of the decomposition theorem
State [CM05, Theorem 2.1.1] and explain its generalisation to the singular case in [CM05, Section 2.3]. Also compare with the classical Hard Lefschetz theorem. Use the decomposition theorem to show that intersection cohomology is a direct summand of the cohomology of a resolution [Dim04, Corollary 5.4.11]. Also discuss the concrete example of a threefold [CM05, Section 2.4].

Talk 9: (15.12.26) Proof part 1: The perverse filtration
Introduce the notion of Whitney stratified maps and explain how every algebraic variety and every algebraic map of algebraic varieties admits a Whitney stratification [CM05, Section 3.2]. Introduce the perverse filtration on cohomology and p-splitness and the weight filtration [CM05, Sections 4.2–4.4]. State and sketch a proof of [CM05, Theorem 4.4.4].

Talk 10: (12.01.27) Proof part 2: The weight filtration and defect of semismallness (Marco)
Introduce the weight filtration [CM05, Theorem 4.5–4.6]. Define the defect of semismallness. Construct the universal hyperplane section and prove that it has smaller defect than the original map. Use this to prove the relative weak Lefschetz theorem [CM05, Section 4.7]. In the end, briefly

explain the inductive procedure of the proof [CM05, Section 2.6] and formulate the other main Hodge-theoretic results [CM05, Theorems 2.1.4 – 2.1.8] that will be needed in the inductive proof.

Talk 11: (19.01.27) Proof part 3: The hard Lefschetz theorem
Prove the decomposition theorem and the hard Lefschetz theorem [CM05, Section 5.1–5.3].

Talk 12: (26.01.27) Proof part 4: Local splitting and semisimplicity
Finish the proof of the main result except for semisimplicity of perverse cohomology [CM05, Section 5.4]. Also cover the first part of the proof of semisimplicity of perverse cohomology in [CM05, Section 6.1].

Talk 13: (02.02.27) Proof part 5: Semisimplicity Complete the proof of semisimplicity of perverse cohomology [CM05, Sections 6.2–6.4].

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