GAUS-AG on *The Kudla program*

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Broadly, the Kudla program is a study of relations between special cycles on Shimura varieties, which come from an automorphic source. This program, initiated by Stephen Kudla close to 30 years ago, touches on numerous threads of ideas and research directions that have seen a lot of progress in recent years. The aim of this seminar is to highlight the key goals of this program, and to survey some of its major results.

Roughly speaking, many of the key results in the Kudla program take the form that a generating series is modular. To sketch a picture, let X be a Shimura variety (say the modular curve). For each integer, m, one associates a subvariety, $Z(m)$ of X. This subvariety is typically an embedding of a smaller Shimura variety. For modular curves, $Z(m)$ is a sum of Heegner points of discriminant m on the modular Jacobian. The goal is to prove that

$$\sum_{m\geq 0}[Z(m)]q^m$$

is a modular form valued in a vector space. This vector space is generally a cohomology group or, even stronger, a Chow group. For modular curves, this is the Gross–Kohnen–Zagier theorem. By applying a functional to a vector-valued modular form, one obtains a classical modular form. One such functional is the degree map, which yields a classical modular form

$$\sum_{m\geq 0}\deg\left([Z(m]\right)q^m.$$

In many cases, the Kudla program relates this modular form to an Eisenstein series.

The special cycles appearing in the Kudla program are of interest because they are, so far, our main tool of approach to the Birch and Swinnerton-Dyer and the Beilinson–Bloch conjectures. In general, it is a great challenge to find families of algebraic cycles. Luckily, Shimura varieties come equipped with a plentiful supply of cycles to study. The key applications of the Kudla program relate special values of L-functions to heights of special cycles, and therefore have important consequences towards the Beilinson–Bloch conjecture.

In this seminar, we will follow in the footsteps of the surveys [10, 11, 5]. The goal is to have a gentle introduction and overview without delving too deep into the technical details. We will flesh out the key constructions, results and proofs of the references mentioned in these surveys. We will begin with the classical theta series and the theta correspondence. We will then move to geometric analogues of the theta correspondence and upgrade the modularity statements from ones valued in cohomology, to Chow groups and finally to arithmetic Chow groups. We end the seminar with a discussion on the recent p-adic analogues to the Kudla program.

Participants are not expected to have already studied Shimura varieties, automorphic forms or Chow groups before. In fact, the setting of orthogonal groups provides a fantastic example based introduction to these topics and their overlap.

Time: We meet on Fridays at 09:15am. Precise dates below.

Place: SR8 (Mathematikon)

Contact: Hazem Hassan, hazem.hassan@iwr.uni-heidelberg.de

Talks

Talk 0: Introduction
Date: 16/10/26, Speaker: Hazem
Reference: [10] [5] [11]

Talk 1: Quadratic spaces and theta functions
Date: 23/10/26, Speaker: Felix
Reference: [5, Section 1,2,3] This talk is an introduction to the basic objects at play in the Kudla program: quadratic spaces and their orthogonal groups. The goal is to briefly go through the definitions of quadratic spaces, their signatures, lattices, genera and orthogonal groups as in Section 2 of [5]. Then move on to discuss theta functions associated with positive definite quadratic spaces: present vector valued theta series, define the Weil representation in this setting, give a sketch of a proof of modularity of the Jacobi theta series

$$\theta(z) = \sum_{n=-\infty}^{\infty} q^{n^2}$$

and present the definite Siegel–Weil theorem [5, Theorem 3.4]).

Talk 2: Classical Theta Correspondences
Date: 06/11/26, Speaker: Will
Reference: [5, Section 4] [3, Section 2.6] [11, Section 2]
This talk will generalize the objects of the previous talk to the setting of indefinite quadratic spaces, where theta functions are not as straightforward. The aim of this talk will be to shift to an adelic language, and to introduce the more general theta kernel for indefinite quadratic spaces as defined in [5, Section 4.2]. Then, specialize to the definite setting to see how to recover the theta series from Talk 1.

Talk 3: Symmetric spaces for Orthogonal groups
Date: 13/11/26, Speaker: Nils
Reference: [10, Sections 1,2] [3, Section 2]
Introduce the symmetric spaces associated to orthogonal groups for all signatures. Discuss the complex structure for quadratic spaces of signature $(n, 2)$. ([3, p. 138]). Define modular forms for orthogonal groups, and Siegel theta series as in [3, Section 2.6]. Give examples of the symmetric spaces in special cases of small dimension such as the upper half-plane for signature (2,1) and the product of upper half-planes for (2,2). Define special cycles in the symmetric spaces, and discuss the example of Heegner points and Hirzebruch–Zagier divisors as in [3, Section 2.7] for signatures (2,1) and (2,2), respectively. Give a sketch of the statements of Kudla–Millson's modularity ([10, Theorem 3.1]).

Talk 4: Kudla–Millson theory
Date: 20/11/26, Speaker: Phillip
Reference: [10, Section 3] [2]
The goal of this talk is to prove the modularity of the Kudla–Millson generating series valued in cohomology. The technical heart of this proof is the construction of certain differential forms that are packaged in a theta series. As the construction of these forms is quite involved, the aim is to give a brief idea of their construction and sketch the properties which they satisfy.

Talk 5: (Arithmetic) Chow groups and intersection pairing
Date: 27/11/26, Speaker: Gebhard
Reference: [8, Chapter 1] [12]
This intermezzo introduces concepts from algebraic geometry that will be required in later talks. In particular, we will need to define Chow groups, the ring structure on Chow groups, the intersection pairing, Chern classes and finally the cycle class map to cohomology. Only the definitions and statements of the results from [8, Chapter 1] on these objects are required and the proofs may be skipped. After that, present a brief motivation of Arakelov theory, the definition of arithmetic Chow groups, the existence of their ring structure, cycle class map and arithmetic heights, following [12].

Talk 6: Borcherds' lifts and modularity in co-dimension 1
Date: 04/12/26, Speaker: Miriam
Reference: [9, Section 4.3] [1]
Sketch the definition of Borcherds' regularized theta lifts and the properties of the resulting meromorphic functions. Then go through Borcherds' proof [1] of the existence of meromorphic differentials and hence the modularity of co-dimension 1 divisors in Chow groups.

Talk 7: Modularity in Chow groups of higher co-dimension.
Date: 11/12/26, Speaker: Andrea
Reference: [13, Chapter 2] [4]
The aim of this talk is to expand Borcherds' result to cycles in higher co-dimension. The relevant modularity result was proved in Zhang's thesis [13] conditional on a convergence condition. This convergence condition follows from the automatic convergence of Bruinier–Raum [4]. This talk will explain Zhang's argument for modularity conditional on the convergence, discuss the statement of automatic convergence of Bruinier–Raum and if time permits go through some of the details of the proof of the main theorem of the main theorem of [4].

Talk 8: Geometric Siegel–Weil theorems and inner product formulae.
Date: 18/12/26, Speaker: Alireza
Reference: [5, Sections 4,5] [11, Sections 2.3, 2.4, 3.6, 3.7][1]
Starting with the classical theta series from [5, Sections 4] and [11, Sections 2.3, 2.4], discuss the Siegel–Weil theorems and the Rallis inner product formula. Then moving to [5, Sections 5] and [11, Sections 3.6, 3.7], upgrade these statements to the geometric setting. Give some remarks on the Weil convergence criterion and on results beyond this criterion.

Talk 9: Kudla's arithmetic generating series conjectures.
Date: 15/01/27, Speaker: Hazem
Reference: [10, Section 2] [5, Section 6] [11, Section 4][1]
The goal of this talk is to upgrade the modularity statements from previous talks to arithmetic Chow groups. Start by discussing, in general (vague) terms, Kudla's conjectures on arithmetic generating series. Then, discuss progress made towards making the conjectures precise and proving some cases of the modularity, arithmetic Siegel–Weil and the arithmetic inner product formula.

[1] [11] deals mostly with unitary groups and not orthogonal groups, but is still helpful since the techniques are not so different and often comments on the orthogonal case as well.

🌶🌶🌶 Talk 10: The p-adic Kudla Program.

Date: 22/01/27, Speaker: Judith

Reference: [7] [6]

The p-adic Kudla program is a more recent development, that seems to mirror the key statements of the Kudla program. The aim of this talk is to survey the results in the p-adic Kudla program and to highlight the facets of similarity with the results from the classical Kudla program.

What 🌶 means

- 🌶 Requiring no advanced background and good resources are available. Great for Masters and PhD students who are new to the subject.
- 🌶🌶 There is either some pre-requisite knowledge, no canonical reference or there is a lot of material to compile. Also requires knowledge of all previous talks. Suitable for PhD students.
- 🌶🌶🌶 Either requiring expert knowledge or a significant time investment in learning a new topic.

References

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[2] Romain Branchereau. The Kudla-Millson form via the Mathai-Quillen formalism. *Canad. J. Math.*, 76(5):1638–1663, 2024.

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[4] Jan Hendrik Bruinier and Martin Westerholt-Raum. Kudla's modularity conjecture and formal Fourier-Jacobi series. *Forum Math. Pi*, 3:e7, 30, 2015.

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[8] David Eisenbud and Joe Harris. *3264 and All That: A Second Course in Algebraic Geometry*. Cambridge University Press, 2016.

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[10] Stephen S. Kudla. *Special Cycles and Derivatives of Eisenstein Series*. Cambridge University Press, 2010.

[11] Chao Li. Geometric and arithmetic theta correspondences, 2025.

[12] Christophe Soulé. *Chapter I: Arithmetic Intersection*, pages 9–36. Springer International Publishing, Cham, 2021.

[13] Wei Zhang. *Modularity of Generating Functions of Special Cycles on Shimura Varieties*. PhD thesis, Columbia University, 2009.