Elementary anabelian varieties are anabelian

GAUS-AG WS26/27

Tim Holzschuh (*tholzschuh@mathi.uni-heidelberg.de*)
Alexander Schmidt (*schmidt@mathi.uni-heidelberg.de*)

The goal of this GAUS AG is to study Magnus Carlson's recent article [Car26], as well as the results of Hoshi [Hos14] on which his results build upon.

Introduction

Grothendieck's "anabelian geometry", as first laid out in his letter [Gro83] to Faltings, is concerned with the question to which extent arithmetic-geometric properties of a variety X over a field k are determined by the *fundamental exact sequence* of étale fundamental groups

$$1 \to \hat{\pi}_1(X_{\bar{k}}, \bar{x}) \to \hat{\pi}_1(X, \bar{x}) \to \mathrm{Gal}_k \to 1.$$

For example, one might ask whether (iso-)morphisms are detected in terms of the fundamental exact sequence. In the case of hyperbolic curves over sub-p-adic fields, this has been proven by Mochizuki in [Moc99]:

Theorem (Mochizuki)**.** Let k be a sub-p-adic field, Y a hyperbolic curve, and T any smooth k-variety. The canonical map $\mathrm{Hom}_k^{\mathrm{dom}}(T, Y) \to \mathrm{Hom}_{\mathrm{Gal}_k}^{\mathrm{op,out}}(\hat{\pi}_1(T), \hat{\pi}_1(Y))$ is a bijection[1].

As a straightforward corollary, one obtains that the fundamental exact sequence is able to fully distinguish isomorphism types of hyperbolic curves:

Corollary (Isomorphism Conjecture)**.** Let X and Y be hyperbolic curves over a sub-p-adic field k. The canonical map $\mathrm{Isom}_k(Y, X) \to \mathrm{Isom}_{\mathrm{Gal}_k}^{\mathrm{out}}(\hat{\pi}_1(Y), \hat{\pi}_1(X))$ is a bijection.

A major open program is to identify anabelian phenomena in higher-dimensions. A natural candidate is what Grothendieck referred to as the class of *elementary anabelian varieties*, i.e. those obtained as successive fibrations of hyperbolic curves:

Definition. Let k be a field. A variety X/k is *elementary anabelian* (also referred to as a *hyperbolic polycurve*) if the structure morphism admits a factorisation

$$X = X_n \to X_{n-1} \to X_{n-2} \to \dots \to X_1 \to \mathrm{Spec}(k)$$

such that each $X_j \to X_{j-1}$ is a relative hyperbolic curve.

Utilising his characterisation of dominant morphisms into hyperbolic curves, Mochizuki managed to prove the Isomorphism Conjecture for hyperbolically fibred surfaces by what

[1] Here, the right-hand side denotes the set of $\hat{\pi}_1(X_{\bar{k}})$-conjugacy classes of *open* homomorphisms $\hat{\pi}_1(T) \to \hat{\pi}_1(Y)$ over Gal_k, while the left-hand side denotes the set of dominant k-morphisms.

looks like the beginning of an inductive process. However, it has proven to be very challenging to make this process into an actual induction beyond low dimensions.

Building on work of Hoshi [Hos14], Carlson recently proved the Isomorphism Conjecture for hyperbolic polycurves in arbitrary dimensions:

Theorem A (Carlson, [Car26, Theorem A]). Let X and Y be hyperbolic polycurves over a sub-p-adic field k. Then the canonical map $\mathrm{Isom}_k(Y,X) \to \mathrm{Isom}^{\mathrm{out}}_{\mathrm{Gal}_k}(\hat{\pi}_1(Y), \hat{\pi}_1(X))$ is a bijection.

He achieves this by proving a characterisation of dominant morphisms into a hyperbolic polycurve Y of arbitrary dimension in terms of maps into its étale homotopy type $\Pi(Y)$. In this higher-dimensional setting, the role of *open* homomorphisms between étale fundamental groups is played by what Carlson calls *stably cohomologically injective* maps between étale homotopy types:

Definition. Let T and Y be connected varieties over a field k and $\varphi\colon\ \Pi(T) \to \Pi(Y)$ a map over $\Pi(k)$.

(1) The map φ is called π_1-*open* if the induced map $\hat{\pi}_1(\varphi)\colon\ \hat{\pi}_1(T) \to \hat{\pi}_1(Y)$ is open.

(2) The map φ is called *cohomologically injective (ci)* if it is π_1-open and the induced map on ℓ-adic étale cohomology

$$\varphi^*\colon\ \mathrm{H}^*_{\text{ét}}(Y_{\bar{k}}, \mathbf{Q}_\ell) = \mathrm{H}^*(\Pi(Y_{\bar{k}}), \mathbf{Q}_\ell) \to \mathrm{H}^*(\Pi(T_{\bar{k}}), \mathbf{Q}_\ell) = \mathrm{H}^*_{\text{ét}}(T_{\bar{k}}, \mathbf{Q}_\ell)$$

is injective for all primes ℓ.

(3) The map φ is called *stably cohomologically injective (sci)* if the induced maps on étale homotopy types on all finite étale covers of Y are cohomologically injective.

(4) We write $[\Pi(T), \Pi(Y)]^{\mathrm{sci}}_{\Pi(k)}$ for the set of homotopy classes of *stably cohomologically injective* maps $\Pi(T) \to \Pi(Y)$ over $\Pi(k)$.

Using his notion of stably cohomologically injective homomorphisms, together with a study of canonical bundles on hyperbolic polycurves, and the Hodge-Tate decomposition of p-adic Hodge theory, Carlson proves:

Theorem B (Carlson, [Car26, Theorem D]). Let Y be a hyperbolic polycurve over a sub-p-adic field k. For any smooth and proper variety T/k that is geometrically connected, the canonical map $\mathrm{Hom}^{\mathrm{dom}}_k(T,Y) \to [\Pi(T), \Pi(Y)]^{\mathrm{sci}}_{\Pi(k)}$ is a bijection.

Since the class of injective maps has excellent formal properties, the above Theorem B is readily seen to be compatible with the inductive procedure outlined by Mochizuki, and thus implies Theorem A.

Time and Place

- Thursdays, 9–11, SR 8, Mathematikon
- First meeting: Oct 15, 2026

Schedule

Overview & first steps with hyperbolic polycurves

In this part, we get an overview of the seminar, distribute topics, and learn some first basic results on hyperbolic polycurves.

Talk 1. Overview. *(Speaker: Tim; Oct 15, 2026).*
Provide an overview of the seminar and distribute the topics.

Talk 2. Hyperbolic polycurves and their covers. *(Speaker: tbd; Oct 15, 2026).* Define hyperbolic polycurves as in [Hos14, Definition 2.1] and prove that, in characteristic 0, they are stable under finite étale covers following [Hos14, Proposition 2.3]. To provide some motivation for the study of hyperbolic polycurves, introduce hyperbolic Artin neighbourhoods [SS16, Definition 6.1] and sketch a proof of the fact that any smooth variety over an infinite perfect field admits a Zariski-basis by hyperbolic Artin neighborhoods, following e.g. [SS16, Lemma 6.3]. Feel free to ignore that they can be even chosen to be *strongly* hyperbolic.

The étale homotopy type of hyperbolic polycurves

In this part, we will introduce the étale homotopy type, study its relation with finite étale covers, and show that hyperbolic (poly-)curves are $\mathrm{K}(\pi, 1)$ schemes.

Talk 3. The étale homotopy type and finite étale covers. *(Speaker: tbd; Oct 22, 2026).*
Recall the definition of the ∞-category of profinite anima (see [HSS, §A.1]) and quickly define the profinite étale homotopy type of a scheme X as the profinite anima $\Pi(X)$ prorepresenting the functor

$$\mathbf{Ani}_\pi \to \mathbf{Ani}, K \mapsto \Gamma_{\text{ét}}(X, \Gamma^* K)$$

of global sections of constant sheaves, see [HSS, §2.1]. Indicate how this universal property essentially boils down to the requirements

$$\hat{\pi}_1(\Pi(X)) = \hat{\pi}_1(X) \quad \text{and} \quad \mathrm{H}^*(\Pi(X), A) = \mathrm{H}^*_{\text{ét}}(X, A)$$

for any locally constant constructible étale sheaf of abelian groups A on X. Explain [HSS, Proposition 3.1] and sketch how this can be used to prove [HSS, Proposition 3.3]. Conclude by using the above to give an alternative way to view [Car26, Construction 2.8] by passing to fibres.

Talk 4. Hyperbolic (poly-)curves are étale $\mathrm{K}(\pi, 1)$-schemes. *(Speaker: tbd; Oct 29, 2026).*
Show that a scheme X is a $\mathrm{K}(\pi, 1)$ in the sense of [Car26, Definition 2.6] if and only if $\Pi(X)$ is a $\mathrm{K}(\pi, 1)$ in the sense of vanishing higher homotopy groups (take inspiration from [HSS, Proposition 3.7]). Use this to show that spectra of fields as well as hyperbolic curves are $\mathrm{K}(\pi, 1)$ (see following [HSS, Proposition 3.8]). Recall Friedlander's fibre sequence (in characteristic 0) [Fri73] (see also [HSS, §5.2]) and deduce that the map $\Pi(X) \to \Pi(S)$ induced by a relative hyperbolic curve is 1-truncated. Conclude that hyperbolic polycurves are $\mathrm{K}(\pi, 1)$.

The étale fundamental group of hyperbolic polycurves

Since we now know that hyperbolic polycurves are $\mathrm{K}(\pi, 1)$, we proceed with a careful study of their fundamental groups.

Talk 5. Strong centre-freeness of $\hat{\pi}_1$. *(Speaker: tbd; Nov 05, 2026).*

The goal of this talk is to explain [Car26, Lemma 2.4 (2)] = [Hos14, Proposition 2.4 (iii)]. Remind the audience that the preceding talk already showed [Car26, Lemma 2.4 (1)] = [Hos14, Proposition 2.4 (i)]. Quickly recall that the geometric étale fundamental group of a hyperbolic curve is a torsion-free, topologically finitely generated profinite group [Tam97, Proposition 1.1 (i), Proposition 1.6]. Show that it is strongly centre-free (aka slim), following either [Nak94, (1.3.1)-(1.3.4)], [MT08, Proposition (1.4)] or by expanding on [Fal98, Lemma 1].

Talk ☺. *(Nov 12, 2026).* Backup slot (if one of the earlier talks needs more time) / discussion session / having a walk / ending earlier.

Talk 6. Elasticity of $\hat{\pi}_1$. *(Speaker: tbd; Nov 19, 2026).*

The goal of this talk is to explain [Car26, Lemma 2.4 (3)] = [Hos14, Proposition 2.4 (iv)]. Recall the notion of *elastic* profinite groups ([Car26, p. 5]), and explain its algebro-geometric motivation as in the paragraph after [Hos14, Remark A.1] in the introduction. Show that the geometric étale fundamental group of a hyperbolic curve is elastic, see [MT08, Theorem 1.5].

Hoshi's polynondegenerate Hom-Conjecture

The goal of this section is to understand the proof of Hoshi's Hom-Conjecture for hyperbolic polycurves in terms of *polynondegenerate* homomorphisms [Hos14, Theorem 3.7].

Talk 7. The factorisation lemma. *(Speaker: tbd; Nov 26, 2026).*

The goal of your talk is to explain the factorisation lemma [Hos14, Lemma 2.9]. Introduce varieties of "LFG-type" as in [Hos14, Definition 2.5] and make sure to explain how they show up in the proof. If time permits, explain [Hos14, Proposition 2.7] and [Hos14, Lemma 2.8] (in this order of priority).

Talk 8. Injectivity of $\mathrm{Hom}_k^{\mathrm{dom}}(T, Y) \to \mathrm{Hom}_{\mathrm{Gal}_k}^{\mathrm{PND,out}}(\hat{\pi}_1(T), \hat{\pi}_1(Y))$. *(Speaker: tbd; Dec 03, 2026).*

As a warmup, recall why dominant morphisms between normal schemes induce open homomorphisms on étale fundamental groups, [Hos14, Lemmata 1.2-1.3]. Prove that for a hyperbolic polycurve Y over a sub-p-adic field k the resulting map

$$\mathrm{Hom}_k^{(\mathrm{dom})}(T, Y) \to \mathrm{Hom}_{\mathrm{Gal}_k}^{(\mathrm{op},)\,\mathrm{out}}(\hat{\pi}_1(T), \hat{\pi}_1(Y))$$

is injective following [Hos14, Proposition 3.2 (i), (ii)]. Feel free to assume the case $n = 1$ if you are short on time. Define *polynondegenerate* homomorphisms [Hos14, Definition 3.6] and, following [Hos14, Theorem 3.7], show that the above injection further factors as

$$\mathrm{Hom}_k^{\mathrm{dom}}(T, Y) \hookrightarrow \mathrm{Hom}_{\mathrm{Gal}_k}^{\mathrm{PND,out}}(\hat{\pi}_1(T), \hat{\pi}_1(Y)) \subset \mathrm{Hom}_{\mathrm{Gal}_k}^{\mathrm{op,out}}(\hat{\pi}_1(T), \hat{\pi}_1(Y)),$$

where the right-hand side denotes the set of polynondegenerate homomorphisms.

Talk 9. Surjectivity of $\mathrm{Hom}_k^{\mathrm{dom}}(T,Y) \to \mathrm{Hom}_{\mathrm{Gal}_k}^{\mathrm{PND,out}}(\hat{\pi}_1(T), \hat{\pi}_1(Y))$. *(Speaker: tbd; Dec 10, 2026).*

Explain [Hos14, Lemma 3.4, Lemma 3.5 (i)] in as much detail as time permits and use it to finish the proof of [Hos14, Theorem 3.7] by showing surjectivity starting from Claim 3.7.B.

Cohomological injectivity

In this part, we set the stage for the proofs of Carlson's main theorems.

Talk 10. Various flavors of cohomological injectivity. *(Speaker: tbd; Dec 17, 2026).*

Cover the content of [Car26, §3.1]. Before giving [Car26, Definition 3.7], recall the basics of the Hodge–Tate decomposition in p-adic Hodge theory by expanding on the paragraph preceding said definition. Bhatt's AWS notes on the subject might be helpful, see e.g. [Bha17, Theorem 1.1.7, Corollary 1.1.11].

Talk ☺. *(Jan 07, 2027)* Backup slot (if one of the earlier talks needs more time) / discussion session / having a walk / ending earlier.

Checkmating hyperbolic polycurves

In this final section, we will prove the main theorems.

Talk 11. Canonical bundles of hyperbolic polycurves. *(Speaker: tbd; Jan 14, 2027).*

The goal of this talk is to show (following [Car26, §2.2]) that (after possibly passing to a finite étale covering) any hyperbolic polycurve X admits a non-zero log-canonical form, see [Car26, Corollary 2.20]. To this end, recall some useful background in log-geometry (log-canonical forms, ...) used in the proof of [Car26, Corollary 2.20]. If you have the time, explain some of the ideas that go into [Sai03, Theorem 1].

Talk 12. Cohomologically injective ⇒ nondegenerate. *(Speaker: tbd; Jan 21, 2027).*

Prove [Car26, Proposition 3.15]. The proof relies on [Car26, Proposition 2.5] and [Car26, Lemma 2.10], which you should state (but only prove if you have extra time).

Talk 13. Opus magnum perfectum est. *(Speaker: tbd; Jan 28, 2027).*

Prove [Car26, Corollary 3.16 & Corollary 3.19] and finally deduce [Car26, Theorem 3.9 & Theorem 3.11].

Talk 🍴. *(Jan 28, 2027)* After the conclusion of a successful seminar, let's go for a hike and have dinner in Heidelberg!

Talk ☺. *(Feb 04, 2027)* Backup slot (if one of the earlier talks needs more time) / discussion session / having a walk / ending earlier.

References

[Bha17] Bhargav Bhatt. "The Hodge-Tate Decomposition via Perfectoid Spaces". Lecture Notes. Arizona Winter School, 2017. URL: https://swc-math.github.io/aws/2017/2017BhattNotes.pdf.

[Car26] Magnus Carlson. *Elementary Anabelian Varieties Are Anabelian*. Apr. 27, 2026. arXiv: 2604.24898. URL: http://arxiv.org/abs/2604.24898.

[Fal98] Gerd Faltings. "Curves and Their Fundamental Groups". In: *Astérisque* (1998). URL: https://smf.emath.fr/publications/expose-bourbaki-840-curves-and-their-fundamental-groups-following-grothendieck.

[Fri73] Eric M. Friedlander. "The Etale Homotopy Theory of a Geometric Fibration". In: *Manuscripta mathematica* (1973). URL: https://eudml.org/doc/154187.

[Gro83] Alexander Grothendieck. *Brief an Faltings*. 1983. URL: https://webusers.imj-prg.fr/~leila.schneps/grothendieckcircle/Letters/GanF.pdf.

[Hos14] Yuichiro Hoshi. "The Grothendieck Conjecture for Hyperbolic Polycurves of Lower Dimension". In: *The University of Tokyo. Journal of Mathematical Sciences* (2014). URL: https://www.ms.u-tokyo.ac.jp/journal/abstract/jms210201.html.

[HSS] Tim Holzschuh, Alexander Schmidt, and Jakob Stix. *Anabelian Geometry in Families*. arXiv: 2608.21232 [math.NT]. URL: http://arxiv.org/abs/2608.21232.

[Moc99] Shinichi Mochizuki. "The Local Pro-p Anabelian Geometry of Curves". In: *Inventiones mathematicae* (Nov. 1, 1999). URL: https://doi.org/10.1007/s002220050381.

[MT08] Shinichi Mochizuki and Akio Tamagawa. "The Algebraic and Anabelian Geometry of Configuration Spaces". In: *Hokkaido Mathematical Journal* (2008).

[Nak94] Hiroaki Nakamura. "Galois Rigidity of Pure Sphere Braid Groups and Profinite Calculus". In: *J. Math. Sci., Tokyo* (1994).

[Sai03] Takeshi Saito. "Log Smooth Extension of a Family of Curves and Semi-Stable Reduction". In: *Journal of Algebraic Geometry* (Dec. 3, 2003). URL: https://www.ams.org/jag/2004-13-02/S1056-3911-03-00338-2/.

[SS16] Alexander Schmidt and Jakob Stix. "Anabelian Geometry with Étale Homotopy Types". In: *Annals of Mathematics. Second Series* (2016).

[Tam97] Akio Tamagawa. "The Grothendieck Conjecture for Affine Curves". In: *Compositio Mathematica* (Aug. 1997). URL: https://www.cambridge.org/core/journals/compositio-mathematica/article/grothendieck-conjecture-for-affine-curves/1992BA14A2D63FA076DB39A34EC45E83.