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Hauptseminar Arithmetische Geometrie
WS 26/27

The 6-Functor Formalism and finiteness/duality for (φ, Γ) -modules

Time: Thursday 11:15 **Place:** hybrid: SR 8 and virtually

The aim of this seminar is to understand first the abstract 6-Functor Formalism following mainly [HM] in order to apply it then within in the work of Mikami [Mikami26] to derive finiteness/duality-results for (φ, Γ) -modules over the Robba ring.

It is well-known (Fargues-Fontaine, Porat, Poyeton) that (φ, Γ) -modules over the Robba ring are related to Γ -equivariant vector bundles on the Fargues-Fontaine curve X . Mikami introduces similarly the "Fargues-Fontaine" solid $\mathcal{D}$ -stack $X_{\mathcal{A}}^{\text{la}}$ over some solid affinoid $\mathbb{Q}_{p, \square}$ -algebra $\mathcal{A}$. This allows him to derive (local) Tate duality (e.g. for $\mathcal{A} = \mathbb{Q}_{p, \square}$) from Poincaré-duality for vector bundles $\mathcal{M}$ of the quotient stack $X_{\mathcal{A}}^{\text{la}}/\Gamma^{\text{la}}$ (considered as objects in some derived category $\mathcal{D}(X_{\mathcal{A}}^{\text{la}}/\Gamma^{\text{la}})$ of quasi-coherent sheaves on $X_{\mathcal{A}}^{\text{la}}/\Gamma^{\text{la}}$):

$$\underline{\text{Hom}}_{\mathcal{A}}(\Gamma(X_{\mathcal{A}}^{\text{la}}/\Gamma^{\text{la}}, \mathcal{M}), \mathcal{A}) \cong \Gamma(X_{\mathcal{A}}^{\text{la}}/\Gamma^{\text{la}}, \underline{\text{Hom}}_{X_{\mathcal{A}}^{\text{la}}/\Gamma^{\text{la}}}(\mathcal{O}_{X_{\mathcal{A}}^{\text{la}}/\Gamma^{\text{la}}} \chi_{\text{cyc}}[2])). \quad (1)$$

If $\mathcal{D}(-)$ belongs to a six functor formalism, then this Poincaré-duality is a formal consequence of the so called $\mathcal{D}$ -smoothness of the structure morphism $X^{\text{la}}/\Gamma^{\text{la}} \rightarrow \text{AnSpec} \mathbb{Q}_{p, \square}$. The required geometric setup, based on solid affinoid spaces and more generally $\mathcal{D}$ -stacks, is taken from the work of Juan Esteban Rodríguez Cargmo (originally [RCa], but extended, refined and with more details elaborated in [RCb]) which in turn is building on the work of Clausen and Scholze, Mann, etc.

In this seminar we follow the inverse route. We firstly want to learn the abstract formalism, then, secondly, we construct the geometric setup in which the quotient stacks $X_{\mathcal{A}}^{\text{la}}/\Gamma^{\text{la}}$ live together with a concrete 6-functor formalism $\mathcal{D}$, before, thirdly, we prove smoothness and, finally, carry over the results to (φ, Γ) -modules over the Robba ring.

Talks

Analytic rings and the (abstract) 6-Functor Formalism

1. **Overview with motivation for to 6-functor formalism** (Anna Blanco Cabanillas ; 15.10.26)
[HM, Introduction up to but not including 1.1, §1.2 up to 1.2.4], see also the 1. Lecture of [Schb]. Motivate the formalism via étale cohomology or topology (without explaining details involving higher categories). Give an overview of the strategy of the seminar; point out that the 6-Functor Formalism is used firstly developing *analytic geometry* and secondly for the proof (1).
2. **Reminder on ∞ -categories and Animation** (Otmar Venjakob; 22.10.26)
[Schb, p. 12-17], [HM, A.1] Provide a general reminder on higher categories and explain symmetric monoidal ∞ -categories via [Schb, Lecture III] including the Straightening/Unstraightening Theorem 3.6. According to [Scha, §11] first motivate the use of animated rings, then recall sifted colimits and the process of animation. We work throughout the seminar with *light condensed mathematics* and take this for granted; as far as notation is concerned we refer to [RCb, §2]. In particular, replace κ -condensed by light condensed everywhere in [Scha, §11]. In particular, we need Lemma 11.8, which says that under certain conditions we have $\text{Cond}(\text{Ani}(\mathcal{C})) = \text{Ani}(\text{Cond}(\mathcal{C}))$. Compare also with [RCb, 4.2.1/2/3]. Define the ∞ -category of animated rings.
3. **Analytic rings** (Marlon Kocher; 29.10.26)
Follow [RCb, §4 and the introduction: Def. 1.2.1] for the definition of analytic animated (associative) rings (but compare also with [Scha, §12]). Define the ∞ -category $\text{AnRing}_{\mathbb{Z}_{\square}}$ of analytic animated rings over $\mathbb{Z}_{\square}$. More precisely, we need from [RCb]: 4.0.1/2, 4.1.1/2/3/4/5/6/7/8/9, give the statements of Prop. 4.1.11/14 without proof, we need Thm. 4.1.13 and section §4.2; omit §4.3, provide the statements of §4.5 and 4.6 (proofs only if instructive and if time permits).
4. **Correspondences and the 6-functor-formalism** (Marlon Kocher; 5.11.26)
Introduce correspondences as in [HM, §2] omitting details (e.g. dualizable objects). Discuss from §3.1, 3.2 in [HM, §2] the definitions and state Prop. 3.1.8 and 3.2.2 together (!) without proof. Provide Definition/Proposition 3.3.2/3 (perhaps without detailed proof), then go on to 3.4.2/6/11. The last theorem is the main result of this talk. Introduce the necessary concepts of sheaves from Appendix A.4.

5. **The kernel category, suave and prim objects** (Name; 12.11.26)

This talk should cover §6.2 from [RCb], details can be found in [HM, §4], from which we need 4.2.4, 4.4.1/5/6/8/9/10/17, 4.5.1/4/5/7/8/9/11/18, 4.6.1, 4.7.4/5. Introduce *dualizable* objects from Appendix B.1. In Lemma 4.8 add the additional equivalence from [Mikami26, Lem. 1.29 + proof]. Also add [Mikami26, the proof of Lem. 1.33, Rem. 1.34]. Prove [Mikami26, Prop. 1.36] using [HM, Prop. D.2.9].

Analytic geometry and the concrete geometric setup (Originally, Mikami refers to [RCa] for the geometric setup he uses. Since [RCb] is an improved (e.g. working in the light condensed world) and more detailed reference we follow the latter source - in practice one has to search and replace Mikami's citations into the first article by places in the second.)

6. **Construction of analytic and solid Huber stacks** (Marco Artusa; 19.11.26)

This talk could introduce analytic stacks as in Section 6.3, and define the category of affinoid rings as in Section 7.1 of [RCb]. In particular, this will require proving some six functor properties of analytic rings and solid Huber rings.

In particular, one has to cover the following: Introduce solid affinoid animated rings [Mikami26, Def. 1.1] (also [RCb, 7.1.1, 7.3.1 combined with Def. 5.5.4]), define the geometric setup $(\mathrm{Aff}_{\mathbb{Z}_{\square}} = \mathrm{Aff}^{\mathrm{Hub}}, E)$ with suitable decomposition (P, I) inducing the six-functor formalism $\mathcal{D}$ [Mikami26, 1.3-1.6, omit exmple 1.4] (compare with [RCb, §6.3], e.g. [RCa, Lem. 3.2.5] corresponds to [RCb, Cor. 6.3.4]), and extend it to $\mathcal{D}$ -stacks [Mikami26, 1.7-1.15], i.e., to the geometric setup $(\mathrm{Shv}_{\mathcal{D}}(\mathrm{Aff}_{\mathbb{Z}_{\square}}), \tilde{E})$ using the $\mathcal{D}$ -topology (compare with [RCb, Def. 6.3.23]); finally we work only with this geometric setup. Provide [Mikami26, 1.43, Rem. 1.44/5], prove [Mikami26, 1.47/48/50/51].

7. **Classical geometric objects as solid stacks: From Huber's analytic adic spaces to solid $\mathcal{D}$ -stacks** (Name; 26.11.26)

This talk could focus in the construction of discrete adic spaces, schemes and adic spaces as "solid Huber stacks" as in Section 5.4 to 5.6 in [RCb], which one only can resume without many proofs.

In particular we need: Introduce animated discrete Huber pairs (R, R^+) [RCb, Def. 5.5.1] and its attached analytic ring $(R, R^+)_{\square}$ [RCb, Def. 5.5.4] - this substitutes [RCa, Def. 2.1.5]. We obtain a colimit preserving, fully faithful functor ([RCb, Prop. 5.5.6] without proof)

$$(-)_{\square} : \mathrm{Ring}^{\mathrm{Hub}} \rightarrow \mathrm{AnRing}$$

which factors through the category $\text{Ring}_{\mathbb{Z}_\square}^{\text{Hub}}$ of solid Huber rings in the sense of [RCb, Def. 7.1.1 (4), Ex. 7.1.4]. Prove [Mikami26, Ex. 1.4, Lem. 1.42] (compare with [RCb, 7.2.3, Proposition 7.2.12], maybe one has to consult also [RCa, Prop. 2.3.2, Lem. 3.2.9]). We obtain a functor

$$\text{AffTateAdic} \rightarrow \text{Aff}_{\mathbb{Z}_\square}(\rightarrow \text{Shv}_{\mathcal{D}}(\text{Aff}_{\mathbb{Z}_\square}))$$

from the classical category of sheafy Tate affinoid adic spaces to the ∞ -category of *solid affinoid spaces* (plus the Yoneda embedding); it extends to a functor from the classical category of sheafy Tate adic spaces

$$\text{TateAdic} \rightarrow \text{Shv}_{\mathcal{D}}(\text{Aff}_{\mathbb{Z}_\square})$$

which maps open immersions to $\mathcal{D}$ -étale morphisms. Explain [Mikami26, §1.3], in particular prove 1.58-62 (Proposition 3.6.13 and Theorem 3.6.15 of [RCa] is Theorem 7.7.21 (+ analytic descent on source and target) in [RCb]). Compare also with [Scha, Prop. 13.16, Rem. 13.17].

The link to Galois representations and (φ, Γ) -modules

The speakers of Talk 8 and Talk 9 are encouraged to coordinate and share the content of their talks with each other if necessary.

8. Locally analytic representations and $h^\dagger$ -analytic representations (Anna Blanco Cabanillas; 3.12.26)

This talk covers [Mikami26, §2.1-2]. The speaker should define the ∞ -category $\text{Rep}_\square(G)$ of representations of a compact p -adic Lie group G (Def 2.7.) and prove, in detail, Proposition 2.34. This result establishes the equivalence of categories

$$\mathcal{D} \left(\text{AnSpec} \mathbf{Q}_{p,\square} / \mathbb{G}^{h^\dagger} \right) \cong \text{Rep}_\square^{h^\dagger}(G)$$

under which, a representation V maps explicitly to the cosimplicial object $\{C^{h^\dagger}(G^n, V)\}_{[n] \in \Delta}$. In the first part of the seminar, we saw that there are natural morphisms p and f associated to any solid $\mathcal{D}$ -stack. In this talk, one should explain how the morphisms $p_h : \text{AnSpec} \mathbf{Q}_{p,\square} \rightarrow \text{AnSpec} \mathbf{Q}_{p,\square} / \mathbb{G}^{h^\dagger}$ and $f_h : \text{AnSpec} \mathbf{Q}_{p,\square} / \mathbb{G}^{h^\dagger} \rightarrow \text{AnSpec} \mathbf{Q}_{p,\square}$ correspond to functors related to locally analytic representations, specifically Proposition 2.36 and 2.39 in [Mikami26].

The speaker should feel free to briefly summarize the part of the talk that recalls the constructions in [RJRC25].

9. **Semilinear representations** (Rustam Steingart; 10.12.26)

This talk covers [Mikami26, §2.3]. Define the category $\mathrm{Rep}_{\mathcal{A}}^{h^\dagger}(G)$ of semi-linear $h^\dagger$ -analytic representations over a solid affinoid animated algebra $\mathcal{A}$ and prove the equivalence

$$\mathcal{D}\left(\mathrm{AnSpec}\mathbf{Q}_{p,\square}/\mathbb{G}^{h^\dagger}\right) \cong \mathrm{Rep}_{\mathcal{A}}^{h^\dagger}(G)$$

in Proposition 2.47. Include Cor. 2.48, Cor. 2.50 and Lemma 2.51, as they are necessary for the subsequent talks.

10. **Tate-Sen method, suaveness and smoothness** (Marvin Schneider; 17.12.26)

Explain [Mikami26, §3.1/2]. Explain the Tate-Sen setting and prove, with as much detail as possible, that the natural morphism

$$p_\infty : \mathrm{AnSpec}(A_\infty, A_\infty^\circ)_{\square}/\mathbb{G}^{la} \rightarrow \mathrm{AnSpec}\mathbf{Q}_{p,\square}$$

is $\mathcal{D}$ -suave (Theorem 3.5) and $\mathcal{D}$ -smooth (Theorem 3.9). Make sure to include Theorem 3.11, as it will be necessary for later talks.

11. **Fargues-Fontaine solid $\mathcal{D}$ -stack and vector bundles** (Rustam Steingart; 21.01.27)

This talk should cover section 3.3 of [Mikami26] from the beginning to Corollary 3.27. Introduce the different coefficient rings at the beginning of [Mikami26, §3.3] and define the Fargues-Fontaine solid $\mathcal{D}$ -stack X^{la} (Def. 3.18). Prove that the morphism $X^{la}/\Gamma^{la} \rightarrow \mathrm{AnSpec}\mathbf{Q}_{p,\square}$ is $\mathcal{D}$ -smooth (Cor. 3.20) and weakly $\mathcal{D}$ -proper (Cor. 3.26).

12. **Cohomology of (φ, Γ) -modules and Poincaré duality** (Marvin Schneider; 28.01.27)

Define vector bundles as in Definition 3.28 [Mikami26] and prove Proposition 3.37. Introduce the Herr-complex [Mikami26, 3.39], state the main Theorem 3.42 (and Cor. 3.42) and prove them by developing from the Appendix as much as possible/necessary.

References

- [HM] Heyer, C. & Mann, L.: *6-Functor Formalisms and Smooth Representations*, arXiv:2410.13038. doi:10.48550/arXiv.2410.13038, <https://arxiv.org/abs/2410.13038>, (2024)

- [RJRC25] Rodrigues Jacinto, J. & Rodríguez Camargo, J. E. : *Solid locally analytic representations*. <https://arxiv.org/pdf/2305.03162>, (2025)
- [RCa] Rodríguez Camargo, J. E.: *The analytic de Rham stack in rigid geometry*. <https://arxiv.org/abs/2401.07738>, (2024)
- [RCb] Rodríguez Camargo, J. E.: *Notes on Solid Geometry*. <https://arxiv.org/abs/2603.03012>, (2026)
- [Scha] Scholze, P.: *Lectures on analytic geometry*. <https://people.mpim-bonn.mpg.de/scholze/Analytic.pdf>, (2019)
- [Schb] Scholze, P.: *Six-Functor Formalisms*. <https://people.mpim-bonn.mpg.de/scholze/SixFunctors.pdf>, (2023)
- [Mikami26] Mikami, Y.: *Finiteness and duality of cohomology of (φ, Γ) -modules and the 6-functor formalism of locally analytic representations*, arXiv:2504.01780. doi:10.48550/arXiv.2504.01780, <https://arxiv.org/abs/2504.01780v2>, (2026)