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Seminar: The K-theory of $\mathbf{Z}/p^n$

AG Venjakob

Summer semester 26

ORGANISATION: We meet every Thursday at 11:15 in SR 8 (Mathematikon).
Online participation is possible - please contact us for the Zoom data.

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Overview

In this seminar we want to study the article "On the K-theory of $\mathbf{Z}/p^n$" by Antieau, Krause and Nikolaus ([AKN22, AKN24]). Using the theory of prismatic cohomology ([BS19]) they are able to compute the higher K-groups of $R := \mathcal{O}_K/\varpi^n$, where $\mathcal{O}_K$ is the ring of integers of a p-adic field and ϖ a uniformizer. While the prime-to-p-part of those K-groups can be calculated with an older result of Quillen about the K-theory of finite fields, the p-part is more involved. The goal of this seminar is to learn the methods of the article and to give proofs of the main results.

The Talks

Talk 1: Introduction, background of K-theory and reduction steps – Marlon (16.4.)

State the main results of the seminar and introduce the relevant definitions of K-theory, e.g. from [Wei13]; also introduce K-groups with ℓ-adic coefficients. Mention the calculation of the prime-to-p part due to Quillen. Discuss reduction steps using [AKN22] and [AKN24, §2.1] via topological cyclic homology $\mathrm{TC}(R, \mathbb{Z}_p)$ to syntomic cohomology $\mathbb{Z}_p(i)(R)$ using Bhatt-Morrow-Scholze's spectral sequence [BMS18, Thm. 1.12 (5)] as black box (but the degeneracy of the spectral sequence and the non-existence of nontrivial extensions will only be covered in Talk 5 [AKN24, Cor. 2.16]). Close with the statement of [AKN22, Thm. 2.1].

Talk 2: Prismatic cohomology relative to δ-rings – Marvin + Rustam (23.4.)

Give a brief overview about (relative) prismatic cohomology. In particular, introduce (relative) δ-rings, prisms, the prismatic site and Breuil-Kisin twists (see [BS19, Introduction up to Def. 1.6, §2-§4] and [AKN23, §2]). Regarding (derived) completions consult [BS19, §1.2] and [BMS18, Def. 4.1].

Talk 3: The prismatic crystal – Marvin + Rustam (30.4.)

With this and the subsequent talk we want to obtain an overview about the article [AKN23]. This talk should cover the necessary ingredients from section §3, which we will use later in the seminar. Recall in particular the p-complete derived ∞-category $D(\mathbb{Z}_p)$ and $\mathbf{E}_\infty$-algebras. From §3 we need everything up to at least Definition 3.7 (Variant 3.6 for $n = 0$ is needed in Def. 6.1). In particular one has to recall/black box derived prismatic cohomology from [BS22, BL]. Lem. 3.14 (using animated rings) is required for the proof of Cor. 6.10, which in turn gives Cor. 7.10, which is Cor. 1.3 from the introduction. Prop. 3.15 corresponds to part (6) of Thm. 1.2. and Ex. 3.21 (a) is crucial with regard to Rem. 6.8/7.9. Use [BMS18, §2.1] to recall the cotangent complex $L_{A/B}$ (needed for Lem. 3.14 and later $L\Omega^j_{R/\mathbb{Z}_p}$ for the proof of Prop. 2.11/ Lem. 2.12 in [AKN24]). We will not discuss the content of §4 and §5, which shows that the site-definition (§2) of relative prismatic cohomology coincides with the crystal approach (§3).

Talk 4: The Nygaard filtration, syntomic cohomology and the prismatic package – Marvin + Rustam (7.5.)

This talk should roughly cover §6, §7 and §8 in [AKN23]; Explain the construction of the Nygaard filtration (Definition 6.6), define relative syntomic complexes resp. cohomology (Definition 7.8) and introduce the prismatic package (§8). Do NOT go into detail; focus on introducing the main notions that we will need for the next talks. Also state Thm. 1.2 (6) and Cor. 1.3 from section §1.

(14.5. – Christi Himmelfahrt)

Talk 5: Crystalline degeneration and relative-to-absolute descent I – Anna (21.5.)

This talk should cover [AKN24, §2.2 - 2.3]. First, discuss how to obtain results for R by comparing to the ring $\mathbb{F}_q[z]/(z^n)$, which under certain conditions have isomorphic associated graded rings. Use [AKN23, Cor. 10.45] without proof. Explain the strategy to work relatively over the Breuil-Kisin prism $W(k)[\![z_0]\!]$ in order to replace objects in the derived category $D(\mathbb{Z}_p)$ by discrete objects, i.e. abelian groups. State Prop. 2.24 and motivate that we need prismatic envelopes for its proof. Summarize the reductions of this section by stating Thm. 2.27. Do NOT discuss Thm. 2.20, which apparently is not needed later.

Talk 6 and 7 should be given by one speaker C and altogether should not last more than 1.5 sessions (= 135 minutes). The aim (of both parts) is to state Prop. 3.33/34 (freeness and finiteness over $W(k)$) and to make them at least plausible, in case you cannot fully prove them.

Talk 6: Prismatic envelopes – Otmar (28.5.)

This talk should cover §3 in [AKN24] (you can use §3.1 as blackbox): Introduce prismatic envelopes and state their basic properties. Explain the δ-ring structure on $\bigoplus N^{\geq i}\widehat{\mathbb{\Delta}}^{(1)}_{R/A}$.

(4.6. – Fronleichnam)

Talk 7: Generators and relations descriptions for the Nygaard filtration – Otmar (11.6.)

The first part of the talk should cover §3.4: Construct the generators for $N^{\geq *}\Delta^{(1)}_{R/A}$ and $\mathrm{gr}^*_N \widehat{\Delta}^{(1)}_{R/A}$ (Construction 3.21 and 3.28) and prove the generators and relations description in Lem. 3.26 and Proposition 3.30. In the second part of the talk, go through §3.5: Specialized to the situation $A = W(k)[\![z_0, \dots, z_s]\!]$, one can describe additive bases, used later to compute the syntomic cohomology of $\mathcal{O}_K / \varpi$, for the filtered pieces $F^{[a,b]}N^{\geq i}\widehat{\Delta}^{(1)}_{R/A}$.

Talk 8: Relative-to-absolute descent II – Chenyi (18.6.)

This talk should cover §4.1 - §4.4 in [AKN24]: Introduce Breuil-Kisin orientations and show how to use them to get rid of the Breuil-Kisin twist in the relative syntomic complex. Explain the cosimplicial diagrams from Lem. 4.25 in (loc. cit.), which connect relative prismatic cohomology with the absolute one. Describe the structure map of the descent complex for $\mathcal{O}_K / \varpi^n$.

Talk 9: Relative-to-absolute descent III – Chenyi (25.6.)

This talk should cover §4.5 - §4.7 in [AKN24]. The goal is to prove Lem. 4.43 (and Cor. 4.44), which gives a quasi-isomorphism from the cochain complexes associated to the relative-to-absolute descent diagrams to a complex consisting of only two non-zero entries and the map ∇ between them. (Omit §4.8 and §4.9: From §4.8 we only need Lem. 4.57 which has to be covered by the next speaker; from §4.9 we only need Lem. 4.64, which is used in Step 7 of the algorithm in §7; similarly, Lem. 4.63 in Step 8, but we shall just black box them! For §7.2, the precision loss of the reductions explained in Remark 4.67, but we will not be able to go into such detail anyway!)

Talk 10: The even vanishing theorem – Jakob (2.7.)

The goal of this talk is to prove the even vanishing theorem ([AKN24, Thm. 5.2]), which tells us that $K_{2i-2}(\mathcal{O}_K / \varpi^n) = 0$ for large enough i. You can use the rather technical results from §5.1 as a blackbox; focus on §5.2 and §5.3 and especially on the proof of the theorem.

Talk 11: An algorithm for syntomic cohomology (optional) – Jon (9.7.)

Go through the algorithm for the computation of the syntomic cohomology ([AKN24, §7.1]). Roughly explain the analysis of the p-adic precision of the algorithm in §7.2, but only go into detail if time permits.

References

[AKN22] Benjamin Antineau, Achim Krause and Thomas Nikolaus, *On the K-theory of* $\mathbf{Z}/p^n$ *– announcement*, arXiv (2022), `https://doi.org/10.48550/arXiv.2204.03420`

[AKN23] Benjamin Antineau, Achim Krause and Thomas Nikolaus, *Prismatic cohomology relative to δ-rings*, arXiv (2023), https://doi.org/10.48550/arXiv.2310.12770

[AKN24] Benjamin Antineau, Achim Krause and Thomas Nikolaus, *On the K-theory of* $\mathbf{Z}/p^n$, arXiv (2024), https://doi.org/10.48550/arXiv.2405.04329

[BMS18] Bhargav Bhatt, Matthew Morrow and Peter Scholze, *Topological Hochschild homology and integral p-adic Hodge theory*, arXiv (2018), https://doi.org/10.48550/arXiv.1802.03261

[BL] Bhargav Bhatt and Jacob Lurie, *Absolute prismatic cohomology*, arXiv (2022), https://doi.org/10.48550/arXiv.2201.06120.

[BS22] Bhargav Bhatt and Peter Scholze, *Prisms and prismatic cohomology*, Ann. of Math. (2) 196 (2022), no. 3, 1135–1275. MR 4502597

[BS19] Bhargav Bhatt and Peter Scholze, *Prisms and Prismatic Cohomology*, arXiv (2019), https://doi.org/10.48550/arXiv.1905.08229

[Wei13] Charles Weibel, *The K-book – an introduction to algebraic K-theory*, AMS (2013), https://sites.math.rutgers.edu/~weibel/Kbook.html

[Qui72] Daniel Quillen, *On the cohomology and K-theory of the general linear groups over a finite field*, Ann. of Math. (1972), https://doi.org/10.2307/1970825