Arboreal GAUS-AG: Specialization of iterated Galois groups

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INTRODUCTION

The goal of this GAUS-AG is to prove the three main theorems of [BGJT25]. The main statement gives a criterion when the image of the arboreal Galois representation associate to a point and a post-critically finite (PCF) map is as large as possible i.e. equals the arithmetic iterated monodromy group.

Let k be a field, k^{sep} a separable closure of k, $\overline{k}$ an algebraic closure and $f \in k(x)$ a separable, rational function of degree $d \geq 2$. We are interested in the dynamics of f and want to study the iterates $f^{\circ n} := f^n := f \circ \ldots \circ f$. Let R be the ramification points of f and $P := \bigcup_n f^n(R)$ the postcritical orbit of f. A rational map is called **post-critically finite (PCF)** if P is finite or in other words the forward orbit of the ramification points is finite. In dynamics, people like to call the ramification points critical points hence the name PCF.
Given $\alpha \in \mathbb{P}^1(k)$ such that $f^n(x) - \alpha$ is separable for all n or equivalently, take $\alpha \in (\mathbb{P}^1 \backslash P)(k)$, then we can built a d-ary complete tree with root α, denoted as $T_\alpha(f)$ in the following way. On level n, we take $f^{-n}(\alpha) = \{\beta \in k^{\mathrm{sep}} \mid f^n(\beta) = \alpha\}$ to be the d^n different vertices and we connect $\beta \in f^{-n}(\alpha)$ with $\gamma \in f^{-(n+1)}(\alpha)$ if $f(\gamma) = \beta$. The absolute Galois group of k acts on the tree $\rho_{\alpha,f}\colon \mathrm{Gal}_k \to \mathrm{Aut}(T_\alpha(f))$ and its image is called an **arboreal Galois group**. One central question in arithmetic dynamics is to understand and classify the image of this action.
We can write the image of $\rho_{\alpha,f}$ as a Galois group (and justifying the name) in the following way. As in [BGJT25], we set for each $n \geq 0$

$$K_{\alpha,n} := k(f^{-n}(\alpha)) \text{ and } G_{\alpha,n} := \mathrm{Gal}(K_{\alpha,n}/k)$$

as well as

$$K_{\alpha,\infty} := \bigcup_n K_{\alpha,n} \text{ and } G_{\alpha,\infty} := \lim_n G_{\alpha,n} \cong \mathrm{Gal}(K_{\alpha,\infty}/k).$$

Then the image of $\rho_{\alpha,f}\colon \mathrm{Gal}_k \to \mathrm{Aut}(T_\alpha(f))$ is isomorphic to $G_{\alpha,\infty}$.

Theorem 1.2 of [BGJT25] states that for a quadratic PCF rational functions f and all $\alpha \in k$ outside a thin set the arboreal Galois group $G_{\alpha,\infty}$ is as large as possible meaning it is equal to the arithmetic iterated monodromy group which we will quickly introduce now.
Let $X := \mathbb{P}^1 \backslash P$ and $\alpha \in X(k)$. Then $\pi_1^{\text{ét}}(X, \alpha)$ acts on $T_\alpha(f)$ and its image is called the arithmetic iterated monodromy group, denoted as G^{arith}. The geometric iterated monodromy group G^{geom} is the image of $\pi_1^{\text{ét}}(X_{\overline{k}}, \alpha)$ inside $\mathrm{Aut}(T_\alpha(f))$. We denote the section corresponding to the k-rational point α as s_α and

take a look at the diagram

$$\begin{array}{ccc} \pi_1(X_{\overline{k}}, \alpha) & \twoheadrightarrow & G^{\mathrm{geom}} \\ \downarrow & & \downarrow \\ \pi_1(X, \alpha) & \twoheadrightarrow & G^{\mathrm{arith}} \\ \downarrow \uparrow s_\alpha & & \downarrow \\ \mathrm{Gal}_k & \xrightarrow{\rho_{\alpha,f}} & \mathrm{Aut}(T_\alpha(f)). \end{array}$$

With the help of the section, we can embed $G_{\alpha,\infty} \cong \mathrm{im}(\rho_{\alpha,f}) \subseteq G^{\mathrm{arith}}$. In the paper, the authors introduce G^{arith} in the following way. Let t be transcendental over k. In other words, we are now considering the generic point of $\mathbb{P}^1_k$ with values in $K = k(t)$. Analogously to before, we define $K_{n,t} \coloneqq K_n \coloneqq k(f^{-n}(t))$, $K_\infty \coloneqq \bigcup K_n$ as well as $G_n = \mathrm{Gal}(K_n/k(t))$ and $G_\infty = \lim_n G_n \cong \mathrm{Gal}(K_\infty/k(t))$. Then $G_\infty \cong G^{\mathrm{arith}}$. Moreover, the authors of [BGJT25] define $k_n \coloneqq \overline{k} \cap K_n$ and $k_\infty = \overline{k} \cap K_\infty$. Then we can also understand the geometric iterated monodromy group as $G^{\mathrm{geom}} \cong \mathrm{Gal}(K_\infty/Kk_\infty)$.

We want to prove the following theorem.

Main Theorem. *Let k be a number field, $f \in k(x)$ a PCF rational function such that* $\mathrm{Gal}(K_1/k_1(t))$ *is a p-group. Then there exists an integer $m \geq 1$ which depends on f and k such that $G_{\alpha,\infty} = G_\infty$ whenever $G_{\alpha,m} = G_m$. In particular, if f is a quadratic rational function this happens for $\alpha \in k$ outside a thin set.*

If the rational function is of the form $f(x) = x^{p^n} + c$ and PCF over a number field k, then the authors can make the criterion for $G_{\alpha,m} = G_m$ more concrete as can be seen in [BGJT25, Theorem 1.4].

DESCRIPTION OF THE TALKS

If not otherwise stated, the numbering is referring to [BGJT25].

Talk 1: Overview of seminar *(May 6 – Leonie Nienhaus)*

Introduce arboreal Galois representations, post critically finite (PCF) functions, the arithmetic and geometric iterated monodromy group. Explain how to embed $G_{\alpha,\infty} \subseteq G_\infty$ for $\alpha \in k$ not strictly post-critical.
State [BGJT25, Question 1.1] and give a quick (selective) overview of the current state of research. Introduce Theorem 1.2, 1.3 and 1.4 of [BGJT25] and give a sketch of the proofs as well as sketch how the seminar talks are linked and used to prove them.

Talk 2: Introduction to arboreal Galois representations *(May 6 – Jakob Stix)*

Introduce and give examples of arboreal Galois representations e.g. where the image is known. For example, one could take a look at section 2.1 of the survey paper [Jon13]. Introduce how to understand the vertices of a rooted tree as letters over a finite alphabet, see section 1.1 of [Nek05] Explain how iterated wreath products work and how to understand the automorphisms of the tree as wreath product, e.g. see section 1.4 of [Nek05]. Explain and proof [JKMT16, Theorem 3.1] which we will later need and which gives a sufficient criterion for the arboreal group being a wreath product. Moreover, explain that G_∞ is the infinite wreath product of G_1 which can be deduced from [JKMT16, Lemma 3.2].
Explain that for a PCF rational function, the image of the arboreal representation within the whole automorphism group of the tree is of infinite index, see for example [Jon13, Theorem 3.1].

Talk 3: Extension of the base field - part I *(June 3 – Ruth Wild)*

Go through the second section of [BGJT25] from Lemma 2.1 until (including) Lemma 2.7.

Talk 4: Extension of the base field - part II *(June 3 – Magnus Carlson)*

Explain the remaining of section 2 of [BGJT25] (Lemma 2.8 until Theorem 2.12). You do not need to prove Lemma 2.9.

Talk 5: Proof of theorems 1.2 and 1.3 *(June 10 – Leonie Nienhaus)*

The goal of this talk is to prove [BGJT25, Theorem 1.3/3.12] and explain how [BGJT25, Theorem 1.2] follows from this. To that end, first go through section 3.1 of [BGJT25]. If there is time, cover section 3.3 of [BGJT25].

Talk 6: Proof of theorem 1.4 *(June 10 – Benjamin Steklov)*

The aim of the last talk is to prove theorem [BGJT25, Theorem 1.4/4.6]. To achieve this, also show [BGJT25, Lemma 4.1] and [BGJT25, Remark 4.2].

ORGANIZATIONAL DETAILS

- Rough schedule:

  14:00 – 15:15: talk 1

  15:15 – 15:45: coffee break

  15:45 – 17:00: talk 2

- Location: Room 711 small, Robert-Mayer-Str. 10, 60325 Frankfurt am Main

- Organizers:
  Leonie Nienhaus (`nienhaus@math.uni-frankfurt.de`)
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BIBLIOGRAPHY

[BGJT25] Robert L. Benedetto, Dragos Ghioca, Jamie Juul, and Thomas J. Tucker. Specializations of iterated Galois groups of PCF rational functions. *Math. Ann.*, 392(1):1031–1050, 2025.

[JKMT16] Jamie Juul, Pär Kurlberg, Kalyani Madhu, and Tom J. Tucker. Wreath products and proportions of periodic points. *Int. Math. Res. Not.*, 2016(13):3944–3969, 2016.

[Jon13] Rafe Jones. Galois representations from pre-image trees: an arboreal survey. *Publications mathématiques de Besançon. Algèbre et théorie des nombres*, pages 107–136, 2013.

[Nek05] V. Nekrashevych. *Self-similar groups.*, volume 117 of *Math. Surv. Monogr.* Providence, RI: American Mathematical Society (AMS), 2005.