Summer 2026
Dates: 07.05, 21.05, 28.05, 11.06.
10-12:30 Uhr, RM 6-8, Raum 309.

GAUS AG
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Version April 16, 2026

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WKB Analysis: From Quadratic Differentials to Stokes matrices

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Abstract We will explore the interplay between the geometry of infinite-area quadratic differentials and the WKB analysis for the associated Schrödinger-type differential equations. This leads to an expression of the wild Riemann-Hilbert map, i.e. Stokes matrices, for SL(2,ℂ)-opers in terms of periods of the quadratic differential (so-called Voros periods). In the final session, we will discuss higher-rank opers and the conjectural extension of the WKB analysis using spectral networks.

1. **Meromorphic differentials and their trajectories** (60 min) 07.05. (Miguel)

   Introduce strata of meromorphic quadratic differentials, the canonical cover, hat-homology lattice and period coordinates [BS15, §2.2+§2.3] [Bai+19]. Discuss the trajectory structure at zeros and poles [BS15, §3.1-3.3] and the associated marked bordered surface [BS15, §6.1]. Show that for a saddle-free GMN differential there is a flat strip decomposition fixing a basis of the hat homology via the crossing/standard saddle connection [BS15, §3.4+3.5]. Explain how the horizontal strip decomposition of a saddle-free GMN differential induces an ideal triangulation of the marked bordered surface [BS15, Lemma 10.1].

   References: [BS15] [Bai+19].

2. $\mathbb{C}P^1$**-structures and Schrödinger equations** (60 min) 07.05. (Nicole)

   Define complex projective structures. Discuss developing map and monodromy. Define the affine structure over the space of quadratic differentials via Schrödinger equation and Schwarzian derivative [LMP09, §1][AB20, §2.1+2.2]. Explain that choosing a square-root of the canonical and a complex projective structure the Schrödinger equation can be globally defined a Riemann surface of positive genus. See [Dum+21, §1.2] for the statement. The local computation can be found in [IN14, §2.1]. Discuss the extension of the results to meromorphic projective structures [AB20, §3].

   References: [LMP09] [AB20].

3. **WKB analysis 1: From Schrödinger equation to the Stokes graph** (60 min) 21.05. (Lukas)

   Introduce the Schrödinger-type equation associated to a quadratic differential on $\mathbb{P}^1$ [Iwa25, §1]. Explain how to construct the formal WKB solution using the Riccati equation Ansatz. Discuss [Iwa25, Example 1.2.] concerning Airy's equation – this will be our running example. Introduce the spectral curve for the quadratic differential, and its turning points (the branch points of the spectral curve). Define the Stokes graph [Iwa25, Definition 1.7.] and describe it in the Airy case.

   References: [Iwa25].

4. **WKB analysis 2: Borel summation and summability of the WKB solution** (60 min) 21.05. (Martin)

   The goal of this talk is to turn the formal WKB solution into analytic solutions away from the Stokes curves. Introduce the notion of Borel summation [Iwa25, §1.2.1.], and explain how it can be used to turn a formal solution into an analytic function. For further background (without parameters) see [PS03, §7.6.]. Illustrate Borel summation for the WKB solution of Airy's equation [Iwa25, Exercise 4]. Explain [Iwa25, Theorem 1.9.] about Borel summability of the WKB solution away from Stokes curves. Sketch the proof, in particular obstructions to Borel summability on Stokes curves, and coming from saddle connections.

   References: [Iwa25] [PS03].

5. **Stokes matrices and Voros periods** (60 min) 28.05. (Konstantin)

   In this talk we compare the exact solutions of the Schrödinger-type equation across different Stokes regions. Explain how to calculate the connection matrices (called Stokes matrices) for Stokes curves emanating from simple zeros [Iwa25, §1.3.1.], and from simple poles [Iwa25, §1.3.3.]. Define Voros periods [Iwa25, Definition 1.15.] and explain how the connection matrices can generally be computed in terms of Voros period integrals on the spectral curve.

   References: [Iwa25].

6. **Gentle introduction to the irregular Riemann-Hilbert correspondence** (60 min) 28.05. (Tianyi)

   The connection matrices from the previous talk are special cases of Stokes matrices, generalized monodromy data attached to connections with higher order poles. Recall the usual Riemann-Hilbert correspondence on a Riemann surface [PS03, Theorem §6.2., Example 6.6.], and sketch how to reconstruct a connection from a local system. Explain what breaks when you allow higher order poles using the exponential function and Airy's equation. Define singular directions for integrable meromorphic connections on a disk with pole order 2 and whose leading term has pairwise distinct eigenvalues [Boa01, §3]. Explain how to define the Stokes matrices [Boa01, Definition 16] using preferred fundamental solutions defined via resummation (use it as black box). For more background on resummation see also [PS03, §7.6.-7.8.]. State the irregular Riemann-Hilbert correspondence [Boa01, Theorem 5] in this case. Explain that the usual monodromy of the connection can be recovered from the Stokes matrices and formal monodromy [Boa01, Lemma 18].

References: [PS03] [Boa01].

7. SL($n, \mathbb{C}$)**-Opers** (60 min) 11.06. (Paul)

   Recall that the global Schrödinger equation on a Riemann surface is equivalent to a SL($2, \mathbb{C}$)-oper [Dum+21, §1.2]. Define SL($n, \mathbb{C}$)-opers [Dum+21, Def. 2.14][Wen16, §4.7]. The main goal is to describe the parametrization of connected components of opers with the SL($n, \mathbb{C}$)-Hitchin base [Wen16, Theorem 4.17]. The construction of the associated SL($n, \mathbb{C}$)-oper is given in [Dum+21, §2.10]. The full theorem is proven in [Wen16, §4.2+§4.3]. Outline the proof.

   References: [Dum+21] [Wen16].

8. SL($n, \mathbb{C}$)**-spectral networks and WKB analysis for opers.** (60 min) 11.06. (Johannes)

   Introduce the spectral cover associated to a point in the meromorphic SL($n, \mathbb{C}$)-Hitchin base. Introduce the root cover, solitons and the WKB spectral network [Nei, §2]. Explain the algorithmic construction of spectral networks [Nei, §2.5]. Note that in general spectral networks are infinite. Introduce [Nei, Conjecture 3.7] and highlight that the techniques of talk 3-5 give a proof for $n = 2$. If times allows, explain the path lifting rule for finite spectral networks as a homotopy invariant way to lift paths along a branched covering [Kin+26, §13.7][HM25, §4][Nei, §4]. Explain how the gluing of the exact solutions of the WKB analysis via Stokes matrices is an example of non-abelianisation [Nei, Prop. 4.4]. Compare to [Kin+26, §13] for a combinatorial view on spectral networks.

   References: [Nei] [Kin+26], original source: [GMN13].

References

[AB20] Dylan G. L. Allegretti and Tom Bridgeland. "The monodromy of meromorphic projective structures". In: *Trans. Am. Math. Soc.* 373.9 (2020), pp. 6321–6367.

[Bai+19] Matt Bainbridge, Dawei Chen, Quentin Gendron, Samuel Grushevsky, and Martin Möller. "Strata of k-differentials". In: *Algebr. Geom.* 6.2 (2019), pp. 196–233.

[Boa01] P. P. Boalch. "Stokes matrices, Poisson Lie groups and Frobenius manifolds." In: *Invent. Math.* 146.3 (2001), pp. 479–506.

[BS15] Tom Bridgeland and Ivan Smith. "Quadratic differentials as stability conditions". In: *Publ. Math., Inst. Hautes Étud. Sci.* 121 (2015), pp. 155–278.

[Dum+21] Olivia Dumitrescu, Laura Fredrickson, Georgios Kydonakis, Rafe Mazzeo, Motohico Mulase, and Andrew Neitzke. "From the Hitchin section to opers through nonabelian Hodge". In: *J. Differ. Geom.* 117.2 (2021), pp. 223–253.

[GMN13] Davide Gaiotto, Gregory W. Moore, and Andrew Neitzke. "Spectral networks". In: *Ann. Henri Poincaré* 14.7 (2013), pp. 1643–1731.

[HM25] Johannes Horn and Martin Möller. *Wall-crossing formulas via spectral networks.* 2025. arXiv: 2508.07727 [math-ag].

[IN14] Kohei Iwaki and Tomoki Nakanishi. "Exact WKB analysis and cluster algebras". In: *J. Phys. A, Math. Theor.* 47.47 (2014). Id/No 474009, p. 98.

[Iwa25] Kohei Iwaki. *Les Houches Lectures on Exact WKB Analysis and Painlevé Equations.* 2025. arXiv: 2512.17599 [math-ph].

[Kin+26] Clarence Kineider, Georgios Kydonakis, Eugen Rogozinnikov, Valdo Tatitscheff, and Alexander Thomas. *Spectral networks. Bridging higher-rank Teichmüller theory and BPS states (to appear).* Vol. 2386. Lect. Notes Math. Cham: Springer, 2026.

[LMP09] Frank Loray and David Marín Pérez. "Projective structures and projective bundles over compact Riemann surfaces". In: *Équations différentielles et singularités. En l'honneur de J. M. Aroca.* Paris: Société Mathématique de France, 2009, pp. 223–252.

[Nei] Andrew Neitzke. *Notes on spectral networks.* URL: https://neitzke.github.io/website/talks/sn-lectures.pdf.

[PS03] Marius van der Put and Michael F. Singer. *Galois theory of linear differential equations.* Vol. 328. Grundlehren Math. Wiss. Berlin: Springer, 2003.

[Wen16] Richard A. Wentworth. "Higgs bundles and local systems on Riemann surfaces". In: *Geometry and quantization of moduli spaces. Based on 4 courses, Barcelona, Spain, March – June 2012.* Basel: Birkhäuser/Springer, 2016, pp. 165–219.