Summer 2026
Dates: 07.05, 21.05, 28.05, 11.06.
10-12:30 Uhr, RM 6-8, Raum 309.

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WKB Analysis: From Quadratic Differentials to Stokes matrices

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Abstract We will explore the interplay between the geometry of infinite-area quadratic differentials and the WKB analysis for the associated Schrödinger-type differential equations. This leads to an expression of the wild Riemann-Hilbert map, i.e. Stokes matrices, for SL(2,$\mathbb{C}$)-opers in terms of periods of the quadratic differential (so-called Voros periods). In the final session, we will discuss higher-rank opers and the conjectural extension of the WKB analysis using spectral networks.

1. **Meromorphic differentials and their trajectories** (60 min) 07.05. (NAME)

   Introduce strata of meromorphic quadratic differentials, the canonical cover, hat-homology lattice and period coordinates [BS15, §2.2+§2.3] [Bai+19]. Discuss the trajectory structure at zeros and poles [BS15, §3.1-3.3] and the associated marked bordered surface [BS15, §6.1]. Show that for a saddle-free GMN differential there is a flat strip decomposition fixing a basis of the hat homology via the crossing/standard saddle connection [BS15, §3.4+3.5]. Explain how the horizontal strip decomposition of a saddle-free GMN differential induces an ideal triangulation of the marked bordered surface [BS15, Lemma 10.1].

   References: [BS15] [Bai+19].

2. $\mathbb{C}P^1$**-structures and Schrödinger equations** (60 min) 07.05. (NAME)

   Define complex projective structures. Discuss developing map and monodromy. Define the affine structure over the space of quadratic differentials via Schrödinger equation and Schwarzian derivative [LMP09, §1][AB20, §2.1+2.2]. Explain that choosing a square-root of the canonical and a complex projective structure the Schrödinger equation can be globally defined a Riemann surface of positive genus. See [Dum+21, §1.2] for the statement. The local computation can be found in [IN14, §2.1]. Discuss the extension of the results to meromorphic projective structures [AB20, §3].

   References: [LMP09] [AB20].

3. **WKB analysis 1: From Schrödinger equation to the Stokes graph** (60 min) 21.05. (NAME)

   Introduce the Schrödinger-type equation associated to a quadratic differential on $\mathbb{P}^1$ [Iwa25, §1]. Explain how to construct the formal WKB solution using the Riccati equation Ansatz. Discuss [Iwa25, Example 1.2.] concerning Airy's equation – this will be our running example. Introduce the spectral curve for the quadratic differential, and its turning points (the branch points of the spectral curve). Define the Stokes graph [Iwa25, Definition 1.7.] and describe it in the Airy case.

   References: [Iwa25].

4. **WKB analysis 2: Borel summation and summability of the WKB solution** (60 min) 21.05. (NAME)

   The goal of this talk is to turn the formal WKB solution into analytic solutions away from the Stokes curves. Introduce the notion of Borel summation [Iwa25, §1.2.1.], and explain how it can be used to turn a formal solution into an analytic function. For further background (without parameters) see [PS03, §7.6.]. Illustrate Borel summation for the WKB solution of Airy's equation [Iwa25, Exercise 4]. Explain [Iwa25, Theorem 1.9.] about Borel summability of the WKB solution away from Stokes curves. Sketch the proof, in particular obstructions to Borel summability on Stokes curves, and coming from saddle connections.

   References: [Iwa25] [PS03].

5. **Stokes matrices and Voros periods** (60 min) 28.05. (NAME)

   In this talk we compare the exact solutions of the Schrödinger-type equation across different Stokes regions. Explain how to calculate the connection matrices (called Stokes matrices) for Stokes curves emanating from simple zeros [Iwa25, §1.3.1.], and from simple poles [Iwa25, §1.3.3.]. Define Voros periods [Iwa25, Definition 1.15.] and explain how the connection matrices can generally be computed in terms of Voros period integrals on the spectral curve.

   References: [Iwa25].

6. **Gentle introduction to the irregular Riemann-Hilbert correspondence** (60 min) 28.05. (NAME)

   The connection matrices from the previous talk are special cases of Stokes matrices, generalized monodromy data attached to connections with higher order poles. Recall the usual Riemann-Hilbert correspondence on a Riemann surface [PS03, Theorem §6.2., Example 6.6.], and sketch how to reconstruct a connection from a local system. Explain what breaks when you allow higher order poles using the exponential function and Airy's equation. Define singular directions for integrable meromorphic connections on a disk with pole order 2 and whose leading term has pairwise distinct eigenvalues [Boa01, §3]. Explain how to define the Stokes matrices [Boa01, Definition 16] using preferred fundamental solutions defined via resummation (use it as black box). For more background on resummation see also [PS03, §7.6.-7.8.]. State the irregular Riemann-Hilbert correspondence [Boa01, Theorem 5] in this case. Explain that the usual monodromy of the connection can be recovered from the Stokes matrices and formal monodromy [Boa01, Lemma 18].

References: [PS03] [Boa01].

7. $\mathrm{SL}(n,\mathbb{C})$**-Opers** (60 min) 11.06. (NAME)

   Recall that the global Schrödinger equation on a Riemann surface is equivalent to a $\mathrm{SL}(2,\mathbb{C})$-oper [Dum+21, §1.2]. Define $\mathrm{SL}(n,\mathbb{C})$-opers [Dum+21, Def. 2.14][Wen16, §4.7]. The main goal is to describe the parametrization of connected components of opers with the $\mathrm{SL}(n,\mathbb{C})$-Hitchin base [Wen16, Theorem 4.17]. The construction of the associated $\mathrm{SL}(n,\mathbb{C})$-oper is given in [Dum+21, §2.10]. The full theorem is proven in [Wen16, §4.2+§4.3]. Outline the proof.

   References: [Dum+21] [Wen16].

8. $\mathrm{SL}(n,\mathbb{C})$**-spectral networks and WKB analysis for opers.** (60 min) 11.06. (NAME)

   Introduce the spectral cover associated to a point in the meromorphic $\mathrm{SL}(n,\mathbb{C})$-Hitchin base. Introduce the root cover, solitons and the WKB spectral network [Nei, §2]. Explain the algorithmic construction of spectral networks [Nei, §2.5]. Note that in general spectral networks are infinite. Introduce [Nei, Conjecture 3.7] and highlight that the techniques of talk 3-5 give a proof for $n = 2$. If times allows, explain the path lifting rule for finite spectral networks as a homotopy invariant way to lift paths along a branched covering [Kin+26, §13.7][HM25, §4][Nei, §4]. Explain how the gluing of the exact solutions of the WKB analysis via Stokes matrices is an example of non-abelianisation [Nei, Prop. 4.4]. Compare to [Kin+26, §13] for a combinatorial view on spectral networks.

   References: [Nei] [Kin+26], original source: [GMN13].

References

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