GAUS-AG on *Symmetric Power Functoriality*

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1. Introduction

Let K be a number field. Langlands functoriality predicts a rather surprising connection between the representation theory of the locally compact group of the adelic points of different reductive groups. In a special case, for any morphism of algebraic groups

$$r : \mathrm{GL}_m \to \mathrm{GL}_n,$$

it predicts a canonical way of transferring automorphic representations of the group $\mathrm{GL}_m(\mathbb{A}_K)$ to automorphic representations of the group $\mathrm{GL}_n(\mathbb{A}_K)$, in a way that is compatible with their associated L-functions and L-parameters.

Automorphic representations can be thought of as generalizations of modular forms. More precisely, if $m = 2$ and $K = \mathbb{Q}$, an automorphic representation of $\mathrm{GL}_2(\mathbb{A}_\mathbb{Q})$ is either generated by a modular eigenform or a harmonic Maass form. Now, consider the algebraic representation

$$\mathrm{sym}^n : \mathrm{GL}_2 \to \mathrm{GL}_{n+1},$$

which is explicitly given on the semisimple elements by the formula

$$\mathrm{diag}(\alpha, \beta) \mapsto \mathrm{diag}(\alpha^n, \alpha^{n-1}\beta, \ldots, \beta^n).$$

Therefore, Langlands functoriality for sym^n predicts that every Hecke eigenform $f \in M_k(N, \chi)$ can be transferred to an automorphic representation of $\mathrm{GL}_{n+1}(\mathbb{A}_\mathbb{Q})$, which we then denote by $\mathrm{sym}^n(f)$. For every embedding $\lambda : \mathbb{Q}(f) \hookrightarrow \overline{\mathbb{Q}_p}$ of the coefficient field $\mathbb{Q}(f)$ of f, if the λ-adic Galois representation attached to f is given by

$$\rho_{f,\lambda} : \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}) \to \mathrm{GL}_2(\overline{\mathbb{Q}_p}),$$

then the λ-adic Galois representation attached to $\mathrm{sym}^n(f)$ should be given by

$$\mathrm{sym}^n(\rho_{f,\lambda}) : \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}) \xrightarrow{\rho_{f,\lambda}} \mathrm{GL}_2(\overline{\mathbb{Q}_p}) \xrightarrow{\mathrm{sym^n}} \mathrm{GL}_{n+1}(\overline{\mathbb{Q}_p}).$$

This particular case of Langlands functoriality is of historical significance, because of its role in the formulation of the Sato–Tate conjecture. In fact, it was observed by Tate that if the symmetric power L-functions of an elliptic curve are well-behaved, one could easily explain some computer calculation made by Sato about the distribution of the Frobenius eigenvalues of that elliptic curve. Langlands functoriality (which was formulated later) explains why one should expect these L-functions to be well-behaved.

Langlands functoriality of sym^n for modular forms has been proved in two papers by Newton and Thorne [NT21a, NT21b]. The first paper proves the result for modular forms of level $N = 1$. The proof has two steps. In the first part of the paper, the authors use p-adic families of automorphic forms to show that if one knows the result for a single eigenform of level $N = 1$ and weight $k \geq 2$, then one can deduce the result for all level 1 eigenforms. In the second part, they use level raising methods and modularity lifting theorems to prove the result for a single example. In this GAUS AG, we will study the first part of the paper [NT21a].

Time: We meet every Thursday at 14:15. Precise dates below.

Place: TBA

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2. Talks

Talk 0: Introduction (Date: 16.04, Speaker: Alireza)
Reference: Introduction of [NT21a].
State the Sato-Tate Conjecture and explain its relation to symmetric power L-functions [MM09, §2-3]. Explain what symmetric power functoriality means and how it fits into Langlands functoriality conjectures. Mention what is known about functoriality of sym^n: [GJ78], [KS02], [Ki03], and [CT17] for small n and [CHT08], [HST10], and [BGG11] for potential automorphy. Explain the main result of [NT21a] and [NT21b] and the main result of Part I of the first paper, which is our main goal for the seminar. Give a high-level overview of the strategy of the proof (see the introduction to Part I: [NT21a, p. 26-27]). Explain the plan for the seminar and distribute the talks.

Talk 1: Unitary Groups (Date: 23.04, Speaker:)
Reference: [NT21a, §1].
Automorphic Background. Cover as much of these as possible: Give a quick introduction to automorphic representations of a reductive group as generalizations of Hecke eigenforms for GL_2. Discuss Flath decomposition and representation theory of p-adic groups. Introduce Jacquet modules and discuss their significance (you can restrict to GL_2 if you want). Discuss local and global Hecke algebras.
Background on Unitary groups. Cover all of these: Define the unitary group G_n [NT21a, (1.0.1)] and compare its automorphic theory to that of GL_n [NT21a, Theorem 1.2 and 1.4]. Discuss their associated Galois representations [NT21a, Corollary 1.3]. Discuss algebraic modular forms and state Lemma 1.23 and 1.24 from [NT21a].

Talk 2: Background on p-adic Modular Forms (Date: 30.04, Speaker: Timo)
Reference: [Em09], [Vo22].
The goal of this talk is to give an introduction to p-adic families of modular forms: Hida families and the Coleman-Mazur eigencurve. You could follow either of the two references. We recommend following [Em09] as the main reference, since the emphasis here is on Hecke eigensystems and Galois representations. Give some examples of congruences of modular forms ([Em09, Example 1.19], [Vo22, §1.3]). Define systems of Hecke eigenvalues ([Em09, Definition 1.14]) and discuss Galois representations attached to them ([Em09, §1.3], [Vo22, p. 4, Remark]). Define Serre's p-adic modular forms ([Em09, Definition 2.10], [Vo22, §2.3]) and mention Galois representations attached to Serre's p-adic eigenforms ([Em09, §2.2]). Quickly discuss Eisenstein families as an example of a p-adic family parametrized by weight ([Em09, §2.3], [Vo22, §2.2]) and then move to Hida families and the eigencurve ([Em09, §2.4], [Vo22, §3.6-3.7])

Talk 3: Background on p-adic Hodge Theory (Date: 08.05 **at 9:15**, Speaker: Miriam)
Reference: [BC09, §2.2].
Quickly review the notions of Hodge-Tate, de Rham, semi-stable and crystalline Galois representations and Hodge-Tate weights. Define (ϕ, Γ)-modules over the Robba ring of an Artinian local $\mathbb{Q}_p$-algebra [BC09, Definition 2.2.1]. Recall the definition of étale (ϕ, Γ)-modules and

state the equivalence of categories with Galois representations [BC09, Proposition 2.2.6 (i)]. Define the cohomology of (ϕ, Γ)-modules [BC09, §2.2.5] and compare it to Galois cohomology [BC09, Proposition 2.2.6 (ii)]. Define the notions of crystalline and de Rham (ϕ, Γ)-modules [BC09, Definition 2.2.10].

Talk 4: Background on Galois Deformation Theory (Date: 21.05, Speaker:)
Reference: [Bö13], [Ch].
Try to cover the following material, but based on the time decide which ones you want to prove: define deformation functors of Galois representations, discuss tangent spaces, and state their representability [Bö13, §1.1-1.4]. Discuss why one defines pseudo-characters (pseudo-representations), define their deformation functor, and discuss representability [Bö13, §2.2-2.5]. Define the character variety and explain [Ch, Theorem 2.2]. Discuss Weil-Deligne representations [Bö13, Theorem 3.9.8] and state [BS06, Proposition 4.1].

Talk 5: Deforming (ϕ, Γ)-Modules (Date: 28.05, Speaker: Benjamin)
Reference: [NT21a, p. 28-33], from §2.3 to Example 2.10.
Define triangulations of (ϕ, Γ)-modules with respect to a parameter. Explain what it means for a parameter to be ordinary, regular, or non-critical. State Lemma 2.4 but you can skip the proof. Define the trianguline deformation functor $\mathcal{D}_{\rho_v, \mathcal{F}_v, \delta_v}$ and state Proposition 2.6. Explain Lemma 2.7 about non-critical triangulations of de Rham representations and prove Lemma 2.8. Finish by specializing to the 2-dimensional case (Example 2.10).

Talk 6: Deforming Global Galois Representations (Date: 11.06, Speaker: Nils)
Reference: [NT21a, p. 33-38], right after Example 2.10 to Proposition 2.15.
Define the global deformation functor with local conditions above p and prove Proposition 2.11 about its tangent space. Define the deformation space of conjugate selfdual pseudocharacters. Explain Lemma 2.12, 2.13, and 2.14 and prove all or some of them depending on available time. Explain and prove Proposition 2.15.

Talk 7: Emerton's Eigenvariety (Date: 25.06, Speaker:)
Reference: [NT21a, p. 39-45], [Em06].
The goal of this talk is to construct the eigenvariety $\mathcal{E}_n$ for the unitary group ([NT21a, §2.18.1]) and study its properties ([NT21a, Proposition 2.22]). Define accessible refinements and discuss Lemma 2.18. Construct the eigenvariety $\mathcal{E}_n$ and discuss Lemmas 2.19 and 2.20. Finally, prove as much of Proposition 2.22 as possible. Our suggestion for this talk is to skip some of the proofs as you see fit and instead provide the necessary background from locally analytic representation theory (in particular, locally analytic Jacquet functor) and completed cohomology for the audience. See also [Kal22, §2.1.5] and [Kal22, §2.2].

Talk 8: $\mathbf{sym^n}$ Functoriality for Unitary Groups (Date: 02.07, Speaker: Judith)
Reference: [NT21a, p. 45-51].
This is a rather technical talk. The goal is to prove [NT21a, Corollary 2.28]. This breaks into two cases: In the regular non-critical case, prove Theorem 2.24 and state the classicality results needed in this case (Lemmas 2.29 and 2.30) but skip their proof. For the ordinary case, prove both the classicality result (Lemma 2.26) and the functoriality (Theorem 2.27).

Talk 9: Descent to the Eigencurve (Date: 09.07, Speaker:)
Reference: [NT21a, p. 54-60].
Review the properties of the Coleman-Mazur eigencurve [NT21a, §2.31]. Prove Lemma 2.34. Deduce Theorem 2.33 from Lemma 2.34 and the results of the last talk.

Talk 10: Eigen-Tennis (Date: 16.07, Speaker: Alireza)
Reference: [NT21a, p. 60-65], [Kal22, §4].
This talk covers the ping-pong argument in [NT21a] and proves the main theorem of the seminar [NT21a, Theorem 3.1]. Explain the result of Buzzard and Kilford about the geometry of the 2-adic eigencurve [NT21a, Theorem 3.2]. Discuss Lemma 3.3 and 3.4. Review the local Langlands correspondence for GL_2 and prove Lemma 3.5. Complete the proof of Theorem 3.1.

References

[BGG11] Barnet-Lamb, Thomas, Toby Gee, and David Geraghty. "The Sato-Tate conjecture for Hilbert modular forms." Journal of the American Mathematical Society 24.2 (2011): 411-469.

[BC09] Bellaıche, Joël, and Gaëtan Chenevier. "Families of Galois representations and Selmer groups." Astérisque 324 (2009): 1-314.

[BS06] Breuil, Christophe, and Peter Schneider. "First steps towards p-adic Langlands functoriality." arXiv preprint math/0603499 (2006).

[Bö13] Böckle, Gebhard. "Deformations of Galois representations." Elliptic curves, Hilbert modular forms and Galois deformations. Basel: Springer Basel, 2013. 21-115.

[Ch] Chenevier, Gaëtan. "Geometric and modular Galois representations, Lecture 5".

[CHT08] Clozel, Laurent, Michael Harris, and Richard Taylor. "Automorphy for some ℓ-adic lifts of automorphic mod ℓ Galois representations." Publications Mathématiques de l'IHÉS 108 (2008): 1-181.

[CT17] Clozel, Laurent, and Jack A. Thorne. "Level-raising and symmetric power functoriality, III." Duke Mathematical Journal 166.2 (2017): 325-402.

[Em06] Emerton, Matthew. "On the interpolation of systems of eigenvalues attached to automorphic Hecke eigenforms." Inventiones mathematicae 164.1 (2006): 1-84.

[Em09] Emerton, Matthew. "p-adic families of modular forms." Séminaire Bourbaki (2009): 2009-2010.

[GJ78] Gelbart, Stephen, and Hervé Jacquet. "A relation between automorphic representations of GL(2) and GL(3) ." Annales scientifiques de l'École Normale Supérieure. Vol. 11. No. 4. 1978.

[HST10] Harris, Michael, Nick Shepherd-Barron, and Richard Taylor. "A family of Calabi-Yau varieties and potential automorphy." Annals of Mathematics (2010): 779-813.

[Kal22] Kallal, Kenz. "p-adic analytic continuation of symmetric power functoriality." (2022).

[Ki03] Kim, Henry. "Functoriality for the exterior square of GL(4) and the symmetric fourth of GL(2)." Journal of the American Mathematical Society 16.1 (2003): 139-183.

[KS02] Kim, Henry H., and Freydoon Shahidi. "Functorial products for GL(2)×GL(3) and the symmetric cube for GL(2)." Annals of mathematics (2002): 837-893.

[MM09] Murty, M. Ram, and V. Kumar Murty. "The Sato-Tate conjecture and generalizations." Current Trends in Science: Platinum Jubilee Special (2009): 639-646.

[NT21a] Newton, James, and Jack A. Thorne. "Symmetric power functoriality for holomorphic modular forms." Publications mathématiques de l'IHÉS 134.1 (2021): 1-116.

[NT21b] Newton, James, and Jack A. Thorne. "Symmetric power functoriality for holomorphic modular forms, II." Publications mathématiques de l'IHÉS 134.1 (2021).

[Vo22] Vonk, Jan. "Overconvergent modular forms."