GAUS-AG: p-adic Simpson correspondence

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Introduction

The classical Corlette-Simpson correspondence describes representations of the fundamental group of a compact Kähler manifold X over $\mathbb{C}$ in terms of Higgs bundles on X. These are pairs (E, θ), where E is a holomorphic vector bundle on X and $\theta\colon E \to E \otimes \Omega^1$ is a map satisfying $\theta \wedge \theta = 0$, called a Higgs field. More precisely, there is a canonical equivalence of categories

$$\left\{\begin{matrix}\text{representations } \pi_1(X) \to \mathrm{GL}(V) \\ \text{on fin. dim. } \mathbb{C}\text{-vector spaces}\end{matrix}\right\} \xrightarrow{\sim} \left\{\begin{matrix}\text{semi-stable Higgs bundles on } X \\ \text{with vanishing Chern classes}\end{matrix}\right\},$$

a version of which recovers the classical Hodge decomposition of the singular cohomology of X (leading to the name "non-abelian Hodge theory"). This correspondence is used in the study of variations of Hodge structures, the geometry of de Rham stacks, and properties of fundamental groups of compact Kähler manifolds, to name just a few.

The p-adic Simpson correspondence seeks to prove a similar theorem in the setting of p-adic geometry. Like for the complex numbers, the history of the subject is extremely rich, and involves the work of many mathematicians over several decades. In this seminar, we want to cover some of Heuer's contributions to the theory, and in particular the proof of the following theorem:

Theorem. *Let X be a proper smooth rigid-analytic variety over a complete algebraically closed extension of $\mathbb{Q}_p$. Then there is an equivalence of categories*

$$\{v\text{-vector bundles on } X\} \xrightarrow{\sim} \{\text{Higgs bundles on } X\}.$$

Here, Higgs bundles are defined as in the complex case, whereas v-bundles don't have an immediate complex analogue and strictly contain the category of (finite-dimensional) representations of the fundamental group. It is still an open question to understand the image of representations under this equivalence, and there are many more variants and applications of this theorem that are actively developed.

Like in the complex case, some of them involve an interpretation of the correspondence in terms of moduli spaces (e.g. [HX]). Understanding the approach via moduli spaces will be the aim of a follow-up seminar next semester.

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We meet on seven Wednesdays during the semester. The meetings consists of two talks that are thematically related, with a coffee break in between. Each talk should be 90-120 minutes long, accommodating for questions during the talk. A short break is encouraged at an appropriate point during the talk. It might happen that the two talks on a given day don't need an equal amount of time, so the speakers are encouraged to coordinate this in advance.

Talk 0: Introduction *(Oct 15, Ruth)*.
A quick ($\approx$ 15 minutes) overview of the proof strategy, and distribution of the remaining talks.

Prerequisites

Talk 1: Rigid and perfectoid spaces *(Oct 15, Benjamin)*.
The goal of this talk is to sketch a proof of almost higher acyclicity of $\mathcal{O}_X^+$ for X an affinoid perfectoid space, as in [Heu24b, Thm. 1.3.41], and to recall all the players involved. So first we need to recall Huber's notion of adic spaces, and the construction of the structure (pre-)sheaf on them. For this, one can follow [SW20, §2,3] or [Hüb24, §2-4]. The aim is to give a broad recollection, and not to go into pathologies (like non-sheafiness). Some very basic examples would be nice, but there is no need to go into the story of the points of the open unit disk over $\mathbb{Z}_p$. Then, we move on to perfectoid algebras and spaces, essentially following [Heu24b, §1.1 - 1.4]. Again, this is a lot to cover, and a good criterion for choosing material is what is needed in the proof of [Heu24b, Thm. 1.3.41]. For getting there, it should suffice to just give a very rough and ready introduction to almost mathematics, not going beyond [SW20, 7.4.1-7.4.3]. The tilting equivalence can also be omitted. There probably is no time to show [Heu24b, Thm.1.3.33], but it would be good to mention it. As an example for perfectoid spaces, it would be nice to see $\mathbb{T}_\infty$, as in [Heu24b, Thm. 1.3.31]. Then, prove [Heu24b, Thm. 1.3.41].

Talk 2: v-topology *(Oct 15, Magnus)*.
This talk develops the necessary notions related to the v-topology required for [Heu22]. Begin by defining the pro-étale site for a rigid space, as in [Heu24b, 1.4.1]. Briefly indicate why the pro-étale site is helpful, perhaps taking inspiration from [Heu24b, 1.4.1], and mention what the pro-étale site of a point looks like for intuition. Mention the natural map $\nu : X_{\text{proét}} \to X_{\text{ét}}$, which will be used throughout the seminar. If time permits, explain what it does in the case $X = \text{Spa}(\mathbb{C}_p)$. Then define the v-topology, starting with some motivation following, for example, [Heu24b, 1.4.2]. Give examples of the structure sheaf in both the v-topology and the pro-étale topology. Then prove the important [SW20, Thm. 17.1.8] in as much detail as possible, recalling the necessary prerequisites. Also mention and if possible, sketch the proof of [KL16, Thm. 3.5.5]. Finally, briefly mention the notion of diamonds [SW20, §8]. If possible, give some motivation for them, and mention their relation to rigid spaces as in [SW20, Prop. 10.2.3].

Hodge-Tate spectral sequences

Talk 3: classical p-adic Hodge-Tate decomposition *(Nov 12, Dmytro)*.
The goal of this talk is to compute $R\nu_*\widehat{\mathcal{O}}_X$, as explained in [Heu24b, Thm. 1.4.36]. The original reference is [Sch12b, Prop. 3.23]. If some necessary notions have not been covered in previous talks, it would be nice to recall them. In particular, it would be good to briefly introduce the notion of the sheaf of differentials on a smooth rigid space. Then, move on to the Hodge-Tate spectral sequence, and how to get the exact sequence

$$0 \longrightarrow \mathrm{H}^1(X, \mathcal{O}_X) \longrightarrow \mathrm{H}^1(X, \widehat{\mathcal{O}}_X) \longrightarrow \mathrm{H}^0(X, \Omega_X(-1)) \longrightarrow 0, \qquad \text{(HT-Add)}$$

for proper X over algebraically closed perfectoid K. For that, it is enough to show the argument for the degeneration of the Hodge-Tate spectral sequence as in [Sch12b, Rem. 3.21]. Further details as in [BMS18, §13] should only be covered if there are no more necessary statements omitted in previous talks that could be explained instead.

Talk 4: Hodge-Tate spectral sequence for $\mathbb{G}_m$ *(Nov 12, Jakob)*.
The goal of this talk is to prove a multiplicative version of (HT-Add), as in [Heu22, Thm. 1.2],

$$0 \longrightarrow \mathrm{Pic}(X) \longrightarrow \mathrm{Pic}(X_v) \xrightarrow{\text{HT log}} \mathrm{H}^0(X, \Omega^1_X)\{-1\} \longrightarrow 0 \qquad \text{(HT-Mul)}$$

under the same assumptions as before. The focus should be to explain some of the additional technical details involved in proving part (i) of this theorem, thus carefully walking through §2 of [Heu22]. §3 can be skipped. Then, finish the proof of the theorem as in §4. It would be nice to use this occasion to learn more about diamantine universal covers, as in §4.3. If necessary, see also [Heu, §2.2] for some statements on tilde-limits.

Rigid analytic groups

Talk 5: Rigid analytic groups 1 *(Nov 19, Magnus)*.
In the later talks, we will need exponentials for commutative rigid analytic groups. The goal of this talk is to introduce them and their Lie algebras, and to establish some basic properties and to see examples. For the definitions, use [Heu, Defn. 3.1] for rigid groups, and [Heu, §3.1] for Lie algebras. Then prove [Far19, Lem. 1, Lem. 5]. We then move towards the exponential, where the goal is to prove [Heu, Prop. 3.5], which gives an exponential

$$\exp : \mathfrak{g}^\circ \hookrightarrow G$$

for $\mathfrak{g}^\circ \subseteq \mathrm{Lie}(G)$ an open $\mathcal{O}_K$-linear subgroup. Then prove [Heu, Cor. 3.8].

Talk 6: Rigid analytic groups 2 *(Nov 19, Ruth)*.
The goal of this talk is to establish a splitting of

$$\log_G : \hat{G} \to \mathfrak{g}$$

for G a commutative rigid group and $\hat{G}$ its subsheaf of topological torsion elements. Define this notion following [Heu24a, §2.2, 2.3], give examples and establish some of its basic properties as in [Heu24a, Prop. 2.14]. Then, establish the existence of the logarithm for $\hat{G}$, as in [HWZ25, Prop. 6.5]. Finally, sketch a proof of [HWZ25, Thm. 6.12], giving the desired splitting.

Relative Picard varieties

Talk 7: Relative Picard varieties *(Dec 10, Katharina)*.
Briefly motivate the notion of Picard varieties, by outlining (without covering the details) how they are used in [Heu24a, §3] to establish the canonical isomorphism

$$\mathrm{Pic}^{\mathrm{tt}}_{X,\mathrm{v}} = \underline{\mathrm{Hom}}(\pi_1^{\text{ét}}(X,x), \mathbb{G}_m)$$

from [Heu24a, Thm. 1.8]. If time permits, briefly explain in what sense $\mathrm{H}^0(X, \Omega^1_X(-1)) \otimes \mathbb{G}_a$ can be regarded as a Hitchin base. The goal of this talk is to then move some to relative Picard varieties. Start by defining the relative Picard variety, as in [Heu25a, Defn. 2.2], and state the main theorem [Heu25a, Thm. 2.4]. Briefly talk about when the relative Picard variety is representable if there is time. Then, shortly elaborate on [Heu25a, Rem. 2.6]. The goal of the rest of the talk is to prove the left-exactness of the sequence in the Thm. 2.4. Cover [Heu25a, §2.3] in as much detail as possible. Some of the nontrivial technical input from [Sch13] and [Sch12a] could be mentioned, in particular, the use of almost purity, c.f. [Sch13, Lem. 3.16, 3.17], [Heu22, §2].

Talk 8: Hodge-Tate sequence for relative Picard varieties *(Dec 10, Jon)*.
Cover the remaining part of §2 of [Heu25a], to prove the right exactness of the sequence in [Heu25a, Thm. 2.4]. Start by covering §2.4, and then use the material developed in this section to prove Thm. 2.4.2, the existence of the partial splitting. Then prove the right exactness following [Heu25a, §2.6]. End by proving (3) and (4) of [Heu25a, Thm. 2.4], as in [Heu25a, §2.7].

Rigidification

Talk 9: Reduction of structure group and rigidification *(Jan 21, Magarethe)*.
The first goal of this talk is to introduce the map $e_{\mathbb{X}}$ and to prove [Heu25a, Thm. 3.2]. In a second step, we move towards rigidification. For that, state and motivate [Heu25a, Defn. 3.6]. Explain how they can be used to show [Heu25a, Prop. 3.7]. Finally, define $\mathcal{R}_{\mathbb{X}}$ as in [Heu25a, Defn. 3.12], and state [Heu25a, Thm. 3.14]. If time allows, give some key ideas of the proof.

Talk 10: Invertible $\mathcal{B}$-modules via the exponential *(Jan 21, Ronald)*.
We start with the description of the Lie algebra of the rigid group $\mathcal{R}^M_{\mathbb{X}}$ that was introduced in the last talk. For that, introduce the sheaf $L\mathcal{R}^M_{\mathbb{X}}$ as in [Heu25a, Defn. 3.15], and sketch the proof of [Heu25a, Lem. 3.17]. The focus of the talk is then to prove [Heu25a, Thm. 3.22] in as much detail as possible. Finally, briefly explain how Thm. 3.22 is used to get a map

$$\{\text{Higgs bundles on } X\} \to \{\text{pro-étale vector bundles on } X\}$$

via $(E,\theta) \mapsto \nu^* E \otimes_{\mathcal{B}_\theta} \mathcal{L}_{\mathcal{B}_\theta}$, for $\mathcal{L}_{\mathcal{B}_\theta}$ from the theorem (this is explained right before the beginning of [Heu25a, §4.1]).

Local correspondence and the canonical Higgs field

Talk 11: The local correspondence *(Feb 4)*.
Define the notion of generalized representations, as in [Fal05, §2] and [Heu, Defn. 2.3]. Explain that generalized $\mathbb{Q}_p$-representations are equivalent to locally free $\mathcal{O}$-modules on X_v in the appropriate setting ([Heu, Prop.2.3]). Then, discuss the local correspondence for small objects ([Heu25a, Thm. 4.4]), following [Fal05, §3] and [Heu25b, Thm. 6.5] (restricted to GL_n). Another reference for this talk could be [He]. If you can, elaborate how this local correspondence compares to the previous calculation of $R^1\nu_*\mathbb{G}_m$. Finally, state and sketch the remaining parts of [Heu25a, §4.1].

Talk 12: Canonical Higgs fields for pro-étale bundles *(Feb 4)*.
Explain the construction of a canonical Higgs field on a pro-étale vector bundle on a smooth rigid space, as in [Heu25a, §4.2].

Proof of the theorem

Talk 13: Proof of the correspondence *(Feb 11)*.
Prove the correspondence as in [Heu25a, §5.1].

Bonus talk: Simpson gerbe *(Feb 11)*.
Give an overview of Bhatt-Zhang's approach to the correspondence via the "Simpson gerbe", as in [Bha].[1]

Technical details

- The meetings start at 2pm and are supposed to end at 6pm. However, it seems likely that some sessions will last a bit longer.
- There will be a coffee break between the talks. Bringing and sharing cake is also encouraged.
- Location: Room 309, Robert-Mayer Strasse 6-10. We can also provide a stream on Zoom.
- The program and scheduling information is available at `ruthwild.gitlab.io`. There is also a mailing list, to which you can be added by sending an e-mail to `wild@math.uni-frankfurt.de`.

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[1] We hope that Bhatt's notes will contain this material by then.

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