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Gaus AG: Higher algebra

AG Venjakob

Wintersemester 2025/2026

ORGANISATION: We meet every Thursday at 11:15 in SR 8 (Mathematikon).
Online participation is possible - please contact us for the Zoom data.

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Overview

The goal of this seminar is to obtain a working knowledge of essential notions in the theory of ∞-categories, higher algebra and spectral algebraic geometry. The selection of topics is done with applications in arithmetic geometry in mind. We are aiming to learn about ∞-derived categories, sheaves (valued in and on ∞-categories), sheaf cohomology, animation and descent results.

The Talks

Talk 1: From categories to ∞-categories – Marvin (23.10.)

References: [7], [8]
Define simplicial sets [8, Definition 1.1.7]. Construct the nerve functor $N\colon \mathrm{Cat} \to \mathrm{sSet}$ [8, Definition 1.1.43], describe its essential image [8, Theorem 1.1.52], as well as its adjoint $h\colon \mathrm{sSet} \to \mathrm{Cat}$ [8, Definition 1.2.5]. Show [8, Corollary 1.2.14] and [8, Proposition 1.2.18]. Define ∞-categories [8, Definition 1.2.15], Kan complexes and ∞-groupoids [8, Definition 1.2.24]. Show [8, Lemma 1.2.19] and mention that the converse statement is also true. Introduce objects, morphisms, subcategories of ∞-categories. [8, 1.2.75 - 86]. See also Section 1 and 2.1. of [7] for a concise overview.

Talk 2: Basic categorical constructions with ∞-categories – Anna (30.10.)

References: [7], [8]
This talk should cover section 2 of [7] with the ultimate goal being the introduction (co)limits in ∞-categories. Start by introducing functors between ∞-categories. Explain how one can think of the set of functors between ∞-categories as a simplicial set and show [7, Lemma 2.2]. If time permits, define Kan, left, right, and inner fibrations and the associated anodyne maps, and explain [8, Theorem 1.3.37 + Corollary 1.3.38], which is then used to show that

the simplicial set of functors between ∞-categories is in fact an ∞-category. Go then on and introduce the mapping space $\mathrm{Map}(x, y)$ between two objects [8, Definition 1.3.47 ff]. Introduce the Join and slice constructions and define the notions of initial/final objects and (co)limits in an ∞-category [7, §2.2 - 2.5].

Talk 3: Interlude: Spectra – Jakob (6.11.)

Following [11, Section 1] Explain the motivation behind the construction of spectra. (It might be helpful to explain the analogy of fibers /cofibres with the analogous notion in abelian categories). Mention in particular, in what sense objects represent cohomology theories. This talk will serve as a motivation for the general notion of stable ∞-categories.

Talk 4: Stable ∞-categories– Nils (13.11.)

References: [3], [7]
The goal of this talk is to explain [3, §1.1, §1.2]. Define stable ∞-categories [3, Definition 1.1.1.9.] and show (without going into to much detail) that the homotopy category of an ∞-category is triangulated [3, Theorem 1.1.2.14.] (for example, one could prove that it is additive and explain how distinguished triangles in the homotopy category are defined). State [3, 1.1.3.1 - 1.1.3.4]. Go then on to [3, §1.2.1] and introduce t-structures on stable ∞-categories. One could also take a look into [7, §5] for an overview.

Talk 5: Derived ∞-categories– Vincent(20.11.)

References: [5]
This talk should give an overview of the theory of derived categories from the persepctive of higher category theory. First, give a reminder on the (classical) Dold-Kan correspondence. Go then on and introduce the derived ∞-category $\mathcal{D}^-(\mathcal{A})$ of an abelian category $\mathcal{A}$ with enough projectives, which is done in [5, §13]. In particular, one should explain the connection to the usual derived category [5, Remark 13.7.]. Go then on and show that $\mathcal{D}^-(\mathcal{A})$ is stable [5, Proposition 13.10].

Talk 6: Sheaves and sheaf cohomology – Marlon (27.11.)

[2, Section 6.2.2 in particular Remark 6.2.2.3] Explain Grothendieck topologies and infinty categorical context. If time permits, mention that, there is a difference between “sheaves on some Grothendieck site” and ∞-topoi, but don’t dwell on this too much. Ellaborate on Remark 6.2.2.3 which asserts that the classical notion of Grothendieck topologies defines a Grothendieck topology on the nerve.
Explain how to compute “sheaf cohomology” using Eilenberg–Maclane objects and how this relates to classical sheaf cohomology (cf. [2, Remark 7.2.2.17]). See also [9] and the nLab article on abelian sheaf cohomology.

Talk 7: Descent vs Hyperdescent – Immanuel (4.12.)

Whether or not an equivalence can be tested on stalks depends on the notion of sheaves used. Following [2, Section 6.5.4.] explain the difference between sheaves and hyper sheaves.

Talk 8: Classical sheaf cohomology as a sheaf – Dahli (11.12.)

(This talk requires good fundmanetal knowledge of the material) It seems to be "well known", that given a sheaf $\mathcal{F}$ w.r.t to some Grothendieck topology on, say the category of Schemes, the functor $\mathbf{R}\Gamma(-,\mathcal{F})$ is a sheaf in the ∞-categorical sense. There seem to be no satsfying accounts of this in the literature and the proof is sketched in [1]. The goal of this talk is to explain this result and fill in as many details as possible.

Talk 9: Animation – Rustam(18.12.)

Follow [10, Section 5.1 up to 5.1.6]. The goal of this talk is to introduce the animation of a category (Subsection 5.1.4). Make sure to cover the examples (5.1.3 and 5.1.6.).

Christmas holidays

Talk 10: Animation II – Jon (8.1.)

Heuristically explain how to pass from animated rings to $\mathbb{E}_\infty$-rings (cf. [4, Section 25.1.2]). Following [10, Section 5.1] explain the notion of Modules over an animated ring and explain what happens if A is a discrete ring. Explain how to construct the tensor product of animated modules and in what sense it coincides with the derived tensor product in the discrete case.

Talk 11: Descent – Max (15.1.)

Explain the notion fpqc, (fppf, étale) topology in this setting. State [4, Corollary D.6.3.3 and Theorem D.6.3.5]. Give an outline of their proofs.

References

[1] Owen Barrett (https://mathoverflow.net/users/37110/owen-barrett), Derived ∞-category of sheaves and ∞-category of sheaves taking values in derived ∞-category, URL (version: 2022-07-31): https://mathoverflow.net/q/427629.

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[3] Lurie, J. Higher Algebra.

[4] Lurie, J. Spectral algebraic geometry .

[5] Lurie, J. Stable Infinity Categories. `https://arxiv.org/abs/math/0608228`

[6] Cnossen, B. Introduction to Higher Algebra. `https://drive.google.com/file/d/1jOFcwhuGCJrdi5xesKpHiz3kehwr6P0a/view`

[7] Groth, M. A short course on ∞-categories. (2015), `https://arxiv.org/abs/1007.2925`

[8] Land, M. Introduction to Infinity-Categories. https://curien.galene.org/notes/Land-infinity-notes.pdf

[9] K.S. Brown, Abstract Homotopy Theory https://ncatlab.org/nlab/files/BrownAbstractHomotopyTheory.pdf

[10] Česnavičius, Kestutis, and Peter Scholze. "Purity for flat cohomology." Annals of Mathematics 199.1 (2024): 51-180.

[11] Malkiewich, C., 2014. The stable homotopy category. Online Lecture Notes. Binghamton University.https://ncatlab.org/nlab/files/MalkiewichStableHomotopyCategory.pdf