

# Gaus AG: Higher algebra

AG Venjakob

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ORGANISATION: We meet every Thursday at 11:15 in SR 8 (Mathematikon).  
Online participation is possible - please contact us for the Zoom data.

CONTACT: Marvin Schneider, mschneider@mathi.uni-heidelberg.de  
Rustam Steingart, rsteingart@mathi.uni-heidelberg.de

## Overview

The goal of this seminar is to obtain a working knowledge of essential notions in the theory of  $\infty$ -categories, higher algebra and spectral algebraic geometry. The selection of topics is done with applications in arithmetic geometry in mind. We are aiming to learn about  $\infty$ -derived categories, sheaves (valued in and on  $\infty$ -categories), sheaf cohomology, animation and descent results.

## The Talks

### Talk 1: From categories to $\infty$ -categories – ??? (23.10.)

References: [7], [8]

Define simplicial sets [8, Definition 1.1.7]. Construct the nerve functor  $N: \text{Cat} \rightarrow \text{sSet}$  [8, Definition 1.1.43], describe its essential image [8, Theorem 1.1.52], as well as its adjoint  $h: \text{sSet} \rightarrow \text{Cat}$  [8, Definition 1.2.5]. Show [8, Corollary 1.2.14] and [8, Proposition 1.2.18]. Define  $\infty$ -categories [8, Definition 1.2.15], Kan complexes and  $\infty$ -groupoids [8, Definition 1.2.24]. Show [8, Lemma 1.2.19] and mention that the converse statement is also true. Introduce objects, morphisms, subcategories of  $\infty$ -categories. [8, 1.2.75 - 86]. See also Section 1 and 2.1. of [7] for a concise overview.

### Talk 2: Basic categorical constructions with $\infty$ -categories – ??? (30.10.)

References: [7], [8]

This talk should cover section 2 of [7] with the ultimate goal being the introduction (co)limits in  $\infty$ -categories. Start by introducing functors between  $\infty$ -categories. Explain how one can think of the set of functors between  $\infty$ -categories as a simplicial set and show [7, Lemma 2.2]. If time permits, define Kan, left, right, and inner fibrations and the associated anodyne maps, and explain [8, Theorem 1.3.37 + Corollary 1.3.38], which is then used to show that

the simplicial set of functors between  $\infty$ -categories is in fact an  $\infty$ -category. Go then on and introduce the mapping space  $\mathrm{Map}(x, y)$  between two objects [8, Definition 1.3.47 ff]. Introduce the Join and slice constructions and define the notions of initial/final objects and (co)limits in an  $\infty$ -category [7, §2.2 - 2.5].

### **Talk 3: Interlude: Spectra (6.11.)**

Following [11, Section 1] Explain the motivation behind the construction of spectra. (It might be helpful to explain the analogy of fibers /cofibres with the analogous notion in abelian categories). Mention in particular, in what sense objects represent cohomology theories. This talk will serve as a motivation for the general notion of stable  $\infty$ -categories.

### **Talk 4: Stable $\infty$ -categories– ??? (13.11.)**

References: [3], [7]

The goal of this talk is to explain [3, §1.1, §1.2]. Define stable  $\infty$ -categories [3, Definition 1.1.1.9.] and show (without going into too much detail) that the homotopy category of an  $\infty$ -category is triangulated [3, Theorem 1.1.2.14.] (for example, one could prove that it is additive and explain how distinguished triangles in the homotopy category are defined). State [3, 1.1.3.1 - 1.1.3.4]. Go then on to [3, §1.2.1] and introduce  $t$ -structures on stable  $\infty$ -categories. One could also take a look into [7, §5] for an overview.

### **Talk 5: Derived $\infty$ -categories– ??? (20.11.)**

References: [5]

This talk should give an overview of the theory of derived categories from the perspective of higher category theory. First, give a reminder on the (classical) Dold-Kan correspondence. Go then on and introduce the derived  $\infty$ -category  $\mathcal{D}^-(\mathcal{A})$  of an abelian category  $\mathcal{A}$  with enough projectives, which is done in [5, §13]. In particular, one should explain the connection to the usual derived category [5, Remark 13.7.]. Go then on and show that  $\mathcal{D}^-(\mathcal{A})$  is stable [5, Proposition 13.10].

### **Talk 6: Sheaves and sheaf cohomology – ??? (27.11.)**

[2, Section 6.2.2 in particular Remark 6.2.2.3] Explain Grothendieck topologies and infinity categorical context. If time permits, mention that, there is a difference between “sheaves on some Grothendieck site” and  $\infty$ -topoi, but don’t dwell on this too much. Elaborate on Remark 6.2.2.3 which asserts that the classical notion of Grothendieck topologies defines a Grothendieck topology on the nerve.

Explain how to compute “sheaf cohomology” using Eilenberg–MacLane objects and how this relates to classical sheaf cohomology (cf. [2, Remark 7.2.2.17]). See also [9] and the nLab article on abelian sheaf cohomology.

### **Talk 7: Descent vs Hyperdescent – ??? (4.12.)**

Whether or not an equivalence can be tested on stalks depends on the notion of sheaves used. Following [2, Section 6.5.4.] explain the difference between sheaves and hyper sheaves.

### **Talk 8: Classical sheaf cohomology as a sheaf – ??? (11.12.)**

(This talk requires good fundamental knowledge of the material) It seems to be “well known”, that given a sheaf  $\mathcal{F}$  w.r.t to some Grothendieck topology on, say the category of Schemes, the functor  $\mathbf{R}\Gamma(-, \mathcal{F})$  is a sheaf in the  $\infty$ -categorical sense. There seem to be no satisfying accounts of this in the literature and the proof is sketched in [1]. The goal of this talk is to explain this result and fill in as many details as possible.

### **Talk 9: Animation – ??? (18.12.)**

Follow [10, Section 5.1 up to 5.1.6]. The goal of this talk is to introduce the animation of a category (Subsection 5.1.4). Make sure to cover the examples (5.1.3 and 5.1.6.).

## **Christmas holidays**

### **Talk 10: Animation II – ??? (8.1.)**

Heuristically explain how to pass from animated rings to  $\mathbb{E}_\infty$ -rings (cf. [4, Section 25.1.2]). Following [10, Section 5.1] explain the notion of Modules over an animated ring and explain what happens if  $A$  is a discrete ring. Explain how to construct the tensor product of animated modules and in what sense it coincides with the derived tensor product in the discrete case.

### **Talk 11: Descent – ??? (15.1.)**

Explain the notion fpqc, (fppf, étale) topology in this setting. State [4, Corollary D.6.3.3 and Theorem D.6.3.5]. Give an outline of their proofs.

## **References**

- [1] Owen Barrett (<https://mathoverflow.net/users/37110/owen-barrett>), Derived  $\infty$ -category of sheaves and  $\infty$ -category of sheaves taking values in derived  $\infty$ -category, URL (version: 2022-07-31): <https://mathoverflow.net/q/427629>.
- [2] Lurie, J. Higher Topos Theory.
- [3] Lurie, J. Higher Algebra.
- [4] Lurie, J. Spectral algebraic geometry .
- [5] Lurie, J. Stable Infinity Categories. <https://arxiv.org/abs/math/0608228>
- [6] Cnossen, B. Introduction to Higher Algebra. <https://drive.google.com/file/d/1j0FcwhuGCJrdi5xesKpHiz3kehwr6P0a/view>
- [7] Groth, M. A short course on  $\infty$ -categories. (2015), <https://arxiv.org/abs/1007.2925>

- [8] Land, M. Introduction to Infinity-Categories. <https://curien.galene.org/notes/Land-infinity-notes.pdf>
- [9] K.S. Brown, Abstract Homotopy Theory <https://ncatlab.org/nlab/files/BrownAbstractHomotopyTheory.pdf>
- [10] Česnavičius, Kestutis, and Peter Scholze. "Purity for flat cohomology." *Annals of Mathematics* 199.1 (2024): 51-180.
- [11] Malkiewich, C., 2014. The stable homotopy category. Online Lecture Notes. Binghamton University. <https://ncatlab.org/nlab/files/MalkiewichStableHomotopyCategory.pdf>