GAUS TECHNISCHE UNIVERSITÄT DARMSTADT

Workshop Linear Algebraic Groups

Darmstadt, 24.07.2025 – 25.07.2025

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We will learn in condensed form about the rich theory of linear algebraic groups. Topics include:

1. Algebraic groups and basic constructions with algebraic groups
2. Groups of multiplicative type, unipotent groups, solvable groups
3. Reductive groups, maximal tori, Borel subgroups, parabolic subgroups
4. Classification of reductive groups via based root data

The level should be suitable for advanced Master's students and generally anyone interested in learning more about algebraic groups.

Location

Room 315
S2|15 Mathematik und Physik
Schlossgartenstraße 7
64289 Darmstadt

Zoom
Meeting ID: 642 6985 0215
Passcode: Smallest six-digit prime number

Organization

- Participants are expected to study the literature on the prerequisites (given below) before the start of the workshop, if needed. They can always contact the organizers should any questions arise. For some important notions, there is background information given below as well.
- The workshop will take place over two days with five talks on the first and four talks on the second day. Each talk should be no longer than 60 minutes and contain at least one proof typical for the topic. We will have a beamer and a blackboard available.
- Funding is available to cover accommodation and travel costs for external speakers. Please contact the organizers by July 1st to apply for funding.
- There will be vouchers to cover the lunch cost at our Mensa.
- There will be a workshop dinner on the first evening at Green Thai. Please let us know if you want to join by July 17th.

Prerequisites

All schemes and varieties will be of finite type over an algebraically closed field. To understand the talks, you should be acquainted with the following notions:

- proper morphisms of schemes (definition, valuative criterion)
- separated morphisms of schemes (definition, valuative criterion)
- group schemes
- (linear) algebraic groups, algebraic subgroups
- fppf-surjectivity of morphisms of algebraic groups, exact sequences and fppf-quotients of algebraic groups
- homogeneous spaces
- normalizers and centralizers of algebraic groups.

We recommend the beginning of [Mil] as well as the corresponding chapters in [GWI] and [GWII] as resources.

Background information

(Linear) algebraic groups

Definition 1. Let k be a field. An *algebraic group* over k is a group scheme, which is of finite type over k.
A morphism of algebraic groups $f : G \to H$ is a morphism of group schemes, where G and H are algebraic groups.

Definition 2. Let k be a field and G an algebraic group over k. An *algebraic subgroup* of G is an algebraic group H over k such that H is a sub-group scheme of G and the inclusion map is a morphism of algebraic groups.

Definition 3. For a field k, a *linear algebraic group* over k is a closed k-sub-group scheme of GL_n for some $n \in \mathbb{N}$.
A morphism of linear algebraic groups $f : G \to H$ is a morphism of group schemes where G and H are linear algebraic groups.

Remark 4. Let k be a field. A group scheme over k is a linear algebraic group if and only if it is affine and of finite type.

Remark 5. Note that group varieties are precisely the smooth algebraic groups and that in characteristic 0 all algebraic groups are smooth.

Quotients of algebraic groups

We roughly follow [GWII, Sections 27.8 and 27.9], where more details can be found.

Definition 6. Let $f\colon X \to Y$ be a morphism of fppf-sheaves over a scheme S (e.g. a morphism of algebraic groups over k). Then f is called *fppf-surjective* if it is an epimorphism in the category of fppf-sheaves, or, equivalently, if for all S-schemes T and for all $y \in Y(T)$ there is an fppf-covering $(g_i\colon T_i \to T)$ and $x_i \in X(T_i)$ such that $f(x_i) = g_i^*(y)$.

Definition 7. A (finite or infinite) sequence of morphisms of group schemes over a scheme S (e.g. of algebraic groups over k)

$$\cdots \longrightarrow G^{i-1} \xrightarrow{f^{i-1}} G^i \xrightarrow{f^i} G^{i+1} \longrightarrow \cdots$$

is *exact* if for all i, the morphism f^{i-1} factors through $\mathrm{Ker}(f^i)$ and the induced morphism $G^{i-1} \to \mathrm{Ker}(f^i)$ is fppf-surjective. In particular, a short sequence of S-group schemes

$$1 \longrightarrow G' \xrightarrow{f} G \xrightarrow{g} G'' \longrightarrow 1$$

is exact if and only if the sequence

$$1 \longrightarrow G'(T) \xrightarrow{f(T)} G(T) \xrightarrow{g(T)} G''(T)$$

is exact for all S-schemes T and g is fppf-surjective.

Definition 8. Let G be an fppf-sheaf of groups on (Sch/S) for some scheme S and let $H \subseteq G$ be an fppf-sheaf of subgroups. The fppf-sheafification of the presheaf

$$(G/H)'\colon (\mathrm{Sch}/S)^{\mathrm{opp}} \longrightarrow (\mathrm{Sets}), \quad T \mapsto G(T)/H(T)$$

is called the *quotient of G by H* and is denoted by G/H. The canonical map $G \to G/H$ is fppf-surjective.

Remark 9. Convince yourself that the following sequence of group schemes over $\mathbb{R}$ is exact:

$$0 \longrightarrow \mathbb{Z}/N\mathbb{Z} \longrightarrow \mathbb{G}_{m,\mathbb{R}} \xrightarrow{(-)\mapsto(-)^N} \mathbb{G}_{m,\mathbb{R}} \longrightarrow 0.$$

Hopf algebras

Further information regarding Hopf algebras can be found in [Med, Chapter 5] and [Mil, Chapter 3].

Definition 10. A *Hopf algebra* is a $k-$algebra $(A, \mathbf{m} : A \otimes A \longrightarrow A)$ together with three $k-$algebra morphisms

$$\Delta : A \longrightarrow A \otimes A,$$
$$S : A \longrightarrow A,$$
$$\epsilon : A \to k,$$

such that the following diagrams commute:[1]

[1] The morphism $k \to A$ is the structure morphism of A.

$$\begin{tikzcd}
A \arrow[r, "\Delta"] \arrow[d, "\Delta"] & A \otimes A \arrow[d, "\Delta\otimes\mathrm{id}"] \\
A \otimes A \arrow[r, "\mathrm{id}\otimes\Delta"] & A \otimes A \otimes A
\end{tikzcd}$$

$$\begin{tikzcd}
A \arrow[r, "\Delta"] \arrow[d, equal] & A \otimes A \arrow[d, "\mathrm{id}\otimes\epsilon"] \\
A \arrow[r, "\simeq"] & A \otimes k
\end{tikzcd}
\qquad
\begin{tikzcd}
A \arrow[r, "\Delta"] \arrow[d, equal] & A \otimes A \arrow[d, "\varepsilon\otimes\mathrm{id}"] \\
A \arrow[r, "\simeq"] & A \otimes k
\end{tikzcd}$$

$$\begin{tikzcd}
A \arrow[r, "\epsilon"] \arrow[d, "\Delta"] & k \arrow[d] \\
A \otimes A \arrow[r, "\mathbf{m}\circ(\mathrm{id}\otimes S)"] & A
\end{tikzcd}
\qquad
\begin{tikzcd}
A \arrow[r, "\epsilon"] \arrow[d, "\Delta"] & k \arrow[d] \\
A \otimes A \arrow[r, "\mathbf{m}\circ(S\otimes\mathrm{id})"] & A
\end{tikzcd}$$

Remark 11. The category of affine algebraic $k-$groups is anti-equivalent to the category of commutative finitely generated Hopf algebras over k.

Program

All schemes and varieties will be of finite type over an algebraically closed field.[2] Throughout all of the talks, we would encourage the speakers to use the language of (group) schemes instead of varieties.

Day 1 — 24.07.2025, 10am – 5pm, Room S2|15 315

10am Talk 1: Representation and Jordan decomposition **(Speaker: Theresa Kaiser)**

Start by explaining the notion of a linear representation of an algebraic group (see [Mil, Section 4.a]). Introduce characters and the associated eigenspaces (see [Mil, 4.g]). Continue by proving [Mil, Cor. 4.24] as well as [Mil, Thm. 4.25].

After introducing the necessary terminology (see [Mil, 9.b] or [Med, 6.1-6.2]), conclude by proving [Mil, Thm. 9.18] or [Med, Thm. 6.2.5] about the Jordan decomposition of algebraic groups.[3]

11am *Small break*

11:15am Talk 2: Diagonalisable Groups **(Speaker: Johann Gramzow)**

The goal is to explain groups of multiplicative type, which are the same as diagonalizable groups because our ground field is algebraically closed ([Mil, 12.a-d]). Define the group $X^*(G)$ of characters of an algebraic group ([Mil, 12.g]). Formulate and prove [Mil, Thm 12.9].[4]

Show that if G is a diagonalizable group, then to endow a finite-dimensional vector space V with the structure of a representation of G is the same as to define an $X^*(G)$-grading on V (see [Mil, 12.13] and [Mil, Thm. 12.23]).

12:15pm *Lunch break*

1:15pm Talk 3: Unipotent Groups **(Speaker: Saskia Kern)**

After introducing the necessary terminology, prove [Mil, Thm. 14.5] and showcase some of its corollaries, in particular [Mil, Cor. 14.6, 14.7, 14.8, 14.12]. Deduce that an algebraic group is unipotent if and only if it has a composition series whose quotients are isomorphic to subgroups of the additive group. (Use that this is the case for the group of unipotent upper triangular matrices, see also [Con, Lem. 20.1.1] and [Mil, Prop. 14.21]. The other implication follows from [Mil, Cor. 14.7].) Explain [Mil, 14.24(a)]. Mention some of the examples in [Con, lecture 15].

2:15pm *Small break*

[2] For most results being separably closed is sufficient.
[3] [Med, Thm. 6.2.5] uses the notion of Hopf algebras for the proof.
[4] Note that since our base field is algebraically closed, you can assume throughout that multiplicative type=diagonalizable, torus=split torus, $\mathrm{Gal}(k_s/k)=0$ etc. You may mention, however, that all of the theory works if k is simply assumed to be separably closed.

2:30pm Talk 4: Solvable Groups and Borel's Fixed Point Theorem **(Speaker: Michelle Klemt)**

Define solvable algebraic groups (see [Mil, Def. 6.26]) and discuss some of their properties (see [Mil, Prop. 6.27]). Introduce the derived group (see [Mil, Def. 6.16] or [Med, Def. 10.1.1]) and the derived series (see [Mil, After Example 6.29] or [Med, Def. 10.1.6]). Then prove [Mil, Prop. 6.30]. State but do not prove [Con, Example 20.1.5].[5] Formulate and, if time permits, prove Borel's fixed point theorem ([Con, 20.2], see also [Mil, Thm. 16.51] for a different proof). Showcase its corollaries [Con, 20.2.2 – 20.2.4].

3:30pm *Coffee & Cake*

4pm Talk 5: Maximal tori, parabolic subgroups, Borel subgroups **(Speaker: Manuel Müller)**

Provide the initial definitions of tori, parabolic subgroups and Borel subgroups of a connected linear algebraic group (see [Mil, Def. 2.11, 17.6, and 17.15], for parabolic subgroups see also [Con, 22.1.9]). Go on to prove [Mil, Thm. 17.9] (note that part a means that B is parabolic), [Mil, Thm. 17.10] and [Mil, Thm. 17.16]. State the normalizer theorem (see [Mil, Thm. 17.48]) and, if time permits, sketch its proof.

If time permits, discuss [Con, Ch. 24.3] (provide [Con, Def. 24.3.2] and state, but do not prove, [Con, Lem. 24.3.3]). Discuss these notions in the context of GL_n (see [Con, Example 24.3.1 and 24.3.5]).

evening *Dinner at Green Thai*

[5] This shows that, under our assumptions, we can drop the split requirement in the subsequent theorems

Day 2 — 25.07.2025, 10am – 4:30pm, Room S2|15 315

10am Talk 6: Semisimple and reductive groups **(Speaker: Jakob Hessel)**

Prove the existence of a largest algebraic subgroup for the relevant properties (see [Mil, 6.g]). Prove [Mil, Lem 6.41]. Define the unipotent radical and discuss the example of GL_n (see [Mil, 19.b] or [Med, Def. 11.1.2]). Then introduce semisimple algebraic groups and reductive algebraic groups (see [Mil, 6.44 and 6.46] or [Med, Def. 11.1.2]). State and, if time permits, prove some properties of reductive groups, e.g. [Con, Lemma 21.5.4, 21.5.5] and [Bor, Cor. 14.11].

Discuss further examples, e.g. $\mathrm{GL}_n, \mathrm{SO}_n, \mathrm{Sp}_{2n}, \mathrm{SL}_n, \mathrm{PGL}_n$, see [Mil, Example 19.19].[6]

Explain but do not prove the classification of reductive groups of rank 1 (see [Mil, Thm. 20.33] or [Con, Thm. 26.1.6]).[7]

11am *Small break*

11:15am Talk 7: Lie algebras **(Speaker: Anna Blanco-Cabanillas)**

Introduce the necessary preliminaries (see [Mil, 10.a] or [Med, 7.1]) to define the Lie algebra of an algebraic group as a vector space (see [Mil, 10.6] or of a linear algebraic group in [Med, Def. 7.2.2]). Discuss the affine case (see [Mil, 10.6]). Consider the examples of GL_n and U_n (see [Mil, 10.7 and 10.8]). Next, discuss and prove some of the basic properties of Lie algebras (see [Mil, 10.c]). Introduce the adjoint representation (see [Mil, 10.20] or [Med, Def. 7.2.10]) and state [Mil, Thm. 10.23]. Provide a proof only if time permits.

Conclude your talk by applying [Mil, Thm. 12.9] (from Talk 2) to the case of the adjoint representation restricted to a maximal torus of a reductive group (see [Mil, After Prop. 21.1]).

12:15pm *Lunch break*

1:15pm Talk 8: Root data **(Speaker: Chirantan Chowdhury)**

Start by providing the initial definitions of a root datum and a system of positive roots (see [Spr, 7.4.1, 7.4.5 (first half)]). Define the Weyl group and prove that it is finite (see [Spr, 7.4.2(c)]). Then define what it means for a root datum to be reduced (last line before [Spr, 7.4.4]).

Show how a root datum can be obtained from a reductive group (see [Mil, Chapter 21.c] and [Med, 11.2]). Provide the (second) definition of the Weyl group (see [Mil, Def. 17.41]) and state [Mil, Prop. 21.1]. Finally, conclude the talk by giving examples of root data as in [Spr, 7.4.7] or [Med, Example 11.2.12].

2:15pm *Small break*

[6] Use the above properties.
[7] This will be needed in Talk 8.

2:30pm Talk 9: Classification of reductive groups **(Speaker: Torsten Wedhorn)**
Explain the classification of reductive groups (see [Spr, Chapter 17]).

3:30pm *Coffee, Cake & Final Discussion*

References

[Bor] A. Borel: Linear Algebraic Groups, Springer Verlag, https://www.math.utah.edu/~ptrapa/math-library/borel/%20Borel-Linear-Algebraic-Groups-1991.pdf

[Bou] N. Bourbaki: Lie groups and Lie algebras, IV – VI, Springer Verlag

[Con] B. Conrad: Course on Linear Algebraic Groups (lecture notes), https://virtualmath1.stanford.edu/~conrad/249BW16Page/handouts/alggroups.pdf

[GWI] U. Görtz, T. Wedhorn: Algebraic Geometry I, Springer Verlag

[GWII] U. Görtz, T. Wedhorn: Algebraic Geometry II, Springer Verlag

[Mil] J. Milne: Algebraic Groups, Cambridge studies in advanced mathematics, https://www.jmilne.org/math/Books/iAG2017.pdf

[Spr] T.A. Springer: Linear Algebraic Groups, Birkhäuser Verlag, https://www.math.utah.edu/~ptrapa/math-library/springer/Springer-Linear-Algebraic%20Groups-1998.pdf

[Med] Tom De Medts: Linear Algebraic Groups, lecture notes, https://algebra.ugent.be/~tdemedts/files/LinearAlgebraicGroups-TomDeMedts.pdf