Goethe-Universität Frankfurt
Institut für Mathematik
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Seminar
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Seminar

Gromov-Witten theory and Real enumerative geometry

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15.-18.Juni

This is preliminary seminar program, providing background for two topics in Gromov-Witten theory with applications to flat surfaces and to ^{1}-homotopic versions of tropical correspondence theorems.

Counting covers The start combines the well-understood counting of torus coverings and the generating function business to see that these are quasi-modular forms. It should acquaint us with the fact that 'everything' in Gromov-Witten theory in such generating functions. At the same time, we will introduce real structures.

Starting after Talk 1 (or maybe Talk 2) we should be able to compute real structures on 'square-tiled' surfaces of small degree. (Here 'square-tiled' should not be taken verbatim, 'origamis' or 'torus covers ramified over one point' is a better terminology here, as the case of squares or rectangles provides only of the possible real structures.)

Moduli space of curves and intersection theory is an interlude topic, for which the program proposes some talks to get everybody up to speed.

Gromov-Witten theory for algebraic geometers There are two approaches to Gromov-Witten theory. The (historically) second, based on ideas of Kontsevich (and Manin), is for counting curves of a fixed cohomology class in an ambient *projective* (i.e. *algebraic*) variety. This requires the notion of Kontsevich's moduli space of stable maps and an early highlight is Kontsevich's recursive formula to count rational curves in the plane. See [KLP18, Section 18] for a brief textbook exposition and [FP97] for details.

Historically the first approach to Gromov-Witten theory is to work in an ambient *symplectic* manifold and to count curves of given class that are psedoholomorphic for some complex structure compatible with the symplectic one. Here, too, there is a notion of a moduli space of stable maps – and different problems to define and compute GW-invariants. We will probably **not touch this aspect**.

During the first meetings we will discuss how to continue:

Branch I: Real Gromov-Witten theory We mention some background and history without aiming to cover this in the seminar talks: The paper [GZ18] seems to be the central reference for real Gromov-Witten theory, i.e. for Lagrangian submanifolds in symplectic varieties with a real structure. Besides the problems for constructing usual Gromov-Witten invariants (compactify the moduli spaces using stable maps, introduce virtual fundamental class) various issues of orientability on various bundles have to be addresse.

The paper [GZ25] also takes the symplectic viewpoint of Gromov-Witten theory, counting Lagrangians in symplectic manifolds. It recalls Kontsevich-Manin's axioms for a Gromov-Witten theory, including 'reconstruction' and 'reduction to primary invariants' and shows that the Real GW defined in [GZ18] satisfy a similar set of 'real' axioms.

The idea to count with the signed Frobenius Schur indicator stems from the paper [GI21] by Georgieva–Ionel. The paper is clearly a carbon copy of [BP08] with additional real involutions (and additional difficulties) added at every appropriate place.

The paper [BP08] defines open Gromov-Witten theory (i.e. for Gromov-Witten invariants for a non-compact space) under the additional condition that the spaces has torus action (here: a splitting rank two vector bundle over a curve) and the fixed point locus is compact. Then the GW-invariants can be *defined* via localization. The paper *solves* the GW-theory, i.e. states glueing laws to reduce the GW-invariants to simpler cases. Via these glueing laws the GW-invariants are functor 2-cobordism category (with values in some R-module). Such functors are equivalent to Frobenius algebras over R. Key to solving the local GW theory is that this Frobenius algebras is semisimple, i.e. a direct sum of one-dimensional algebras.

Concretely: Without knowing much of the reason, we can work with a proposed real count, namely with the signed Frobenius Schur indicator and see what we get, following . This is pure representation theory of symmetric groups and some topology (of Klein surfaces). To complete the references, [IZ18] might be one of the latest papers trying to approach the real count on covers of the projective line (where 'polynomials' means with a fully ramified point at infinity).

Branch II: The DR-cycle One of the approaches to compute the fundamental class of strata of differentials goes via this algebraic theory of Gromov-Witten invariants with rubber target, in fact via the double ramification (DR)-cycle and Pixton's formula. Using the space of twisted differentials and Holmes-Schmitt's diamond space avoids the use of virtual fundamental classes for most of the definitions. The proof in [JPPZ17] and [BHPSS23] uses the full GW-machinery, but technicalities can be black boxed.

1. **Real structures in low genus** (30.04.25, 60 min) Andrei Bud

   Classify the real structures on the torus, the quotients being the annulus, the Möbius strip or the Klein bottle, distinguished by the number of boundary components being two, one ore zero. Possibly classify real structures on $\mathbb{P}^1$, too.

   The result is stated in [AG71, Section I.9]. This certainly requires the notion of Klein surface (vs. Riemann surface). Probably we want briefly the definition of meromorphic functions on Klein surfaces, as prerequsite for the notion of morphism [AG71, Section I.4]. It certainly requires the definition of various canonical double covers, the complex double [AG71, Section I.6] and for comparison also the orientation double. Complete the classification, by stating for which $\tau \in \mathbb{H}$ which real structures are possible.

Literatur: [AG71].

2. **Counting coverings** (30.04.25, 60 min) Jeonghoon So

   Prove the Burnside Lemma (aka Frobenius formula) [LZ04, A.1.10] (building on A.1.9) for the nunber of coverings with given ramification profile in terms of characters of finite groups.

   Deduce from there and generating functionology that ´[CMZ18, Section 6.1] the generating series for the number of covers without unramified components is given by q-brackets. Show how to get from there by inclusion-exclusion to the number of connected covers. Give examples in small complexity.

   Literatur: [LZ04, Appendix], [CMZ18] .

3. **Shifted Symmetric functions and quasimodular forms**
   (07.05.25, 60 min) Nicole Müller

   The goal is to prove that the generating functions for counting connected covers are quasimodular forms. This is for the simplest case of simple branche points the work of Kaneko-Zagier, for the case of general branch points the Bloch-Okounkov theorem. The key notion is that of a shifted symmetric function.

   The proof in [Zag16] is much more readable and condensed than the original version of Bloch-Okounkov. Maybe the summary in [CMZ18, Section 8] is helpful.

   Literatur: [CMZ18], [Zag16].

4. **Quick start to $\overline{M}_{g,n}$ and its intersection theory**
   (07.05.25, 60 min) Lukas Schneider

   Provide a working introduction (without proofs) to the definition of the Deligne-Mumford compactification $\overline{M}_{g,n}$ and its standard line bundles. Introduce the λ-, κ-, and ψ-classes. Introduce clutching and forgetful maps. Provide examples for the simplest evaluations of these classes on small dimensional moduli spaces. State the main relations between these classes. The suggested main source is Zvonkine's introduction. This material is also contained in [ACG11], mainly in Chapter 17.

   Literatur: [Zvo12], [ACG11].

5. **Cohomology of the moduli space of pointed rational curves**
   (14.05.25, 60 min) N.N.

   For the moduli space of pointed rational curves (and contrary to moduli spaces of higher genus) the whole Chow ring is understood thanks to the work of Keel. Possible sources are [KLP18, Section 15] and [ACG11, Chapter 17.7]

There is a similar (but technically much more involved) description of the cohomology of the compactified moduli space of real rational curves with conjugate Marked Points in [CGZ23]. Maybe one can mention the precise definition ofthis space and state some results. Technical details are far beyond the scope.

Literatur: [KLP18, Section 15], [ACG11], [CGZ23].

6. **Moduli of stable maps** (14.05.25, 60 min) N.N.

As motivation give Kontsevich's count of rational curves in the plane. This is smoothly presented with limited details in [KLP18, Section 18]. The main tool is the moduli space of stable maps. Many more details are given [FP97]. Skip most of them. We certainly can take on faith the existence of the space, which is a GIT-and-glueing construction. The one important thing to know is the dimension formula (Theorem 2 and Section 5.2 of [FP97]) This involves integral over β of the first Chern class of the tangent bundle. The formula explains (a bit) why Calabi-Yau threefolds (and more generally target varieties with trivial canonical bundle) work best for Gromow-Witten theory.

Quantum Cohomology and the Associativity Equation can now be used to prove Kontsevich's result. To prove this, use the description of the cohomology of $\overline{M}_{0,n}$ from the previous talk. More details and more applications are in Section 9 of [FP97]

Literatur: [KLP18, Section 18], [FP97].

7. **Signed real Hurwitz numbers** (60 min) N.N.

This talk aims to *work with* the signed Frobenius Schur indicator introduced in [GI21], without asking why it has been introduced there.

Introduce the signed Frobenius Schur indicator, probably focussing on the group $G = S_n$ exclusively. Indicate how this is related to properties of the underlying Irrep [Gui23, Lemma 7]. Under the restriction that the underlying real surface has no boundary (Equation (4.1)), the signed count is defined in Definition 14. It seems to depend on an 'admissible class', but Lemma 15 shows that it does not. Prove Theorem 1, which is now a computation similar to the computation with the Burnside Lemma in Talk 2.

The examples in [Gui23, Section 4.5] seem very useful. Can we do an example of a small degree torus cover?

The signed Real Hurwitz numbers are topological, hence they should satisfy recursive formulas using degeneration, similar to how many Gromov-Witten theories are solved. These are presented in Section 5. Probably this should not be part of the talk for time reasons.

Section 6 claims to drop the restriction made in Equation (4.1). It looks inconclusive. Maybe discuss what is done and what is missing.

Literatur: [Gui23].

8. **Optional: More on the topology of real curves** $\overline{M}_{g,n}$ (60 min) N.N.

   Structure of the moduli space of real curves inside the Deligne-Mumford compactification.

   Potential side project: what happens if we replace the moduli space of curves with strata in this paper(s)?

   Literatur: [GH81], [SS89].

9. **Towards Real GW theory:** (60 min) N.N.

   The symplectic side, the fundamental paper [GZ18] and its companion papers are hard and quite a bit away from the algebraic background. Understanding the content of the open real GW theory [GI21] looks doable, but certainly requires understanding the (complex) open GW theory in [BP08] first. That paper is quite readable. To be discussed in detail.

   Literatur: [BP08], [GI21].

10. **GW-branch II The DR-cycle: II.1 twisted differentials** (two talks of 60 min) N.N.

    As one motivation: consider the moduli space of twisted differentials from [FP18] as one motivation to 'compactify' strata of differentials, namely taking the naive compactification plus extra components entirely in the boundary. Putting suitable weights on these extra components, the appendix states the (now proven) Conjecture A that this weighted sum is equal to Pixton's formula.

    On the other hand, there is the double ramification (DR) cycle, and its k-twisted version, defined as the locus where the bundles $\omega^{\otimes k}(-\sum m_i z_i)$ is trivial. This make sense over the interior of $\overline{M}_{g,n}$ and over some boundary strata (compact type), but not all of them, see [Pix23] for the history. One has to go the a blowup of $\overline{M}_{g,n}$, the diamond-space of [HS21] to define the DR cycle by this strategy. Sketch the deformation theory in [HS21] to show that the weighted version of the moduli spaces of twisted differentials

    For a motivation how Pixton come up with such a formula the best source is [Pix23].

    Literatur: [FP18], [HS21], [Pix23].

11. **II.2 Relative stable maps** (60 min) N.N.

    This prepares for the proof of Pixton's formula: The basic source is Jun Li's notes "Lecture Notes on Relative GW-lnvariants" (available e.g. from the ICTP website). We need the entirely algebraic side, not to be confused with the arXiv notes by An-Min Li, Li Sheng with the same title, that work in the symplectic category.

12. **II.3 The proof of the DR-cycle formula** (60 min) N.N.

    This combines the main argument in [JPPZ17] (for the untwisted DR-cycles) and maybe [BHPSS23], which does the k-twisted DR-cycle. If we do this, we already need the background about relative stable maps. Then it is not so difficult. There are three ingredients: Chiodo's formula (that could be taken as black box), the localization formula for GW-invariants and how to compute (with) $\mathbb{C}^*$-fixed points, and how to deal with the limit as the weighting-index r goes to infinity.

    Literatur: [JPPZ17], [BHPSS23].

Literatur

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