GAUS AG: Motivic Spectra

April 4, 2025

The goal of this seminar is to study the category of (non-$\mathbb{A}^1$-invariant) motivic spectra MS_S as defined in [AI22, AHI25]. We will start with the definition and its basic properties, and then devote a big part of the seminar to the study of (non-$\mathbb{A}^1$-invariant) algebraic cobordism $\mathrm{MGL} \in \mathrm{MS}_S$. The last part of the seminar will be about Atiyah duality in MS_S, the reference for this is [AHI24]. In the references, everything is proven for derived schemes. We will ignore this added generality and focus on the classical case.

Organization Each talk will be assigned a speaker. However, a talk need *not* last only 90 minutes! Prepare as much material as you deem appropriate, and if you run out of time, just continue the next week. As a rough guideline, an easy talk might take 90 minutes, a medium one 120, and a hard one 180 (i.e. two full sessions). We anticipate to reach about talk 9 by the end of the semester. The date in the talk description below is the **expected** date *when the talk starts* (but not necessarily ends). Be aware that these dates may change, depending on how fast or slow we progress through the talks.

Usually, the seminar will be on Monday 10-12. There are three exceptions:

- In the week of the 21.4. the seminar will be on **Tuesday, 22.4. 10-12**.
- In the week of the 19.5. the seminar will be on **Monday, 19.5. 8-10**.
- In the week of the 9.6. the seminar will be on **Tuesday, 10.6. 10-12**.

Talks

Talk 1: Andreas Gieringer **Easy – 14.4.**

Talk 2: Timo Weiß **Hard – 22.4.**

Talk 3: Timon Tausendpfund **Medium – 5.5.**

Talk 4: Xiaowen Dong **Easy – 12.5.**

Talk 5: Klaus Mattis **Medium – 19.5.**

Talk 6: Lorenzo Mantovani **Hard – 2.6.**

Talk 7: Julie Bannwart **Medium – 16.6.**

Talk 8: Philipp Schleiß **Medium – 23.6.**

Talk 9: Anton Engelmann **Medium – 30.6.**

Talk 1: Definition of motivic spectra (Andreas Gieringer) **Easy – 14.4.**

Define elementary and smooth blowup excision [AHI25, Definition 2.1], and prove that Nisnevich-locally they agree [AHI25, Proposition 2.2]. Explain how to invert and object c in a presentable symmetric monoidal ∞-category $\mathcal{C}$, i.e. explain [AI22, Proposition 1.3.14]. Note that in general, this is not given by the c-telescope $\mathrm{colim}_{Pr^L}(\mathcal{C} \xrightarrow{-\otimes c} \mathcal{C} \to \dots)$. End the talk by giving the definition of MS_S, as in the beginning of [AHI25, Section 4]. If time permits, explain the basic functoriality of MS_S.

Talk 2: Thom-spaces (Timo Weiß) **Hard – 22.4.**

Define the Thom space functor as in the beginning of [AHI25, Section 3]. Spend the rest of the talk proving (as much as possible about) [AHI25, Corollary 3.7].

Talk 3: Properties of motivic spectra (Timon Tausendpfund) **Medium – 5.5.**

In the first half of the talk, prove [AHI25, Theorem 4.1]. Then state and deduce $\mathbb{P}^1$-homotopy invariance and weighted $\mathbb{A}^1$-invariance, [AHI25, Corollaries 4.4 and 4.8]. State

[AHI25, Corollary 4.11], and use it to prove the Bass fundamental theorem [AHI25, Proposition 4.12]. Discuss with the next speaker about possibly different allocations of material between this and the next talk.

Talk 4: A geometric model for the stack of vector bundles (Xiaowen Dong) **Easy – 12.5.**

Discuss with the previous speaker about possibly different allocations of material between this and the previous talk. Prove [AHI25, Corollary 4.11]. Prove [AHI25, Corollary 4.13 and Proposition 4.14]. In the second half of the talk, explain the geometric model for the stack of vector bundles, i.e. prove [AHI25, Proposition 5.1, Lemma 5.2 and Theorem 5.3].

Talk 5: Orientations of motivic spectra (Klaus Mattis) **Medium – 19.5.**

Explain oriented motivic spectra as in [AHI25, Section 6]. Define projective bundle formula, and prove that elementary blowup excision is equivalent to the projective bundle formula [AI22, Lemma 3.3.5]. Explain the Thom isomorphism, and compute the cohomology of the stack of vector bundles [AHI25, Theorem 6.3]. Define Thom-orientations and prove [AHI25, Proposition 6.8].

Talk 6: Snaith Theorem for algebraic K-theory (Lorenzo Mantovani) **Hard – 2.6.**

Define the motivic spectrum $\mathrm{KGL} \in \mathrm{MS}_S$, and prove as detailed as possible [AI22, Theorem 5.3.3].

Talk 7: Algebraic Cobordism (Julie Bannwart) **Medium – 16.6.**

Explain the definition of $\mathrm{MGL} \in \mathrm{MS}_S$, as outlined in [AHI25, Section 7], in particular, explain the representation via the canonical bundles on Grasmannians, cf. [AHI25, Proposition 7.1]. Explain the Thom-isomorphism on cohomology of [AHI25, Proposition 7.4]. Then prove that maps of homotopy ring spectra out of MGL correspond to orientations, cf. [AHI25, Theorem 7.5].

Talk 8: Algebraic Conner-Floyd isomorphism (Philipp Schleiß) **Medium – 23.6.**

The goal of this talk is to prove [AHI25, Theorem 8.11]. For this, introduce the different notions of cohomology theories [AHI25, Definition 8.1]. Then proceed by proving the different universal properties of K-theory and algebraic cobordism, cf. [AHI25, Propositions 8.5, 8.8 and 8.10]. Then combine these results to obtain the Conner-Floyd isomorphism [AHI25, Theorem 8.11].

Talk 9: Snaith Theorem for periodic algebraic cobordism (Anton Engelmann) **Medium – 30.6.**

Define PMGL as in the beginning of [AHI25, Section 7], and quickly prove the properties analogous to MGL: [AHI25, Remark 7.2, Corollary 7.8, Corollary 8.6]. Spend the rest of the talk proving [AHI25, Theorem 9.3].

Talk 10: The Gysin transformation and the geometric evaluation map **Hard**

State the existence of the Gysin map, and its properties (in the incoherent version, [AHI24, Remark 2.4]). Define the Gysin transformation [AHI24, Construction 2.6]. If there is time, state some of the properties of the Gysin transformation [AHI24, Proposition 2.9]. In the last part of the talk, define the geometric evaluation map [AHI24, Construction 3.1], and prove as much as possible of [AHI24, Proposition 3.8]. Discuss with the next speaker about possible different allocations of the material of this and the next talk.

Talk 11: Duals of projective space **Hard**

Discuss with the previous speaker about possible different allocations of the material of this and the previous talk. Explain the Gysin null sequence and prove that it is a cofiber sequence in favourable cases, [AHI24, Construction 3.3 and Proposition 3.4]. Spend the majority of the talk proving as much as possible of [AHI24, Theorem 4.8].

Talk 12: Ambidexterity and Atiyah duality, I **Medium**

Define the exceptional functors $f^!$ and $f_!$ for smooth morphisms, as in [AHI24, Section 5]. Sketch proofs of the base change and projection formulas. Then give the construction of the trace map [AHI24, Construction 5.3], and use it to define MS-ambidextrous functors.

Prove [AHI24, Lemma 5.5 and Proposition 5.7]. Discuss with the next speaker about possible different allocations of the material of this and the next talk.

Talk 13: Ambidexterity and Atiyah duality, II **Medium**

Discuss with the previous speaker about possible different allocations of the material of this and the previous talk. Prove as detailed as possible [AHI24, Theorem 5.9]. Reap the fruits of our hard work and use this theorem to prove [AHI24, Lemma 5.12 - Corollary 5.15].

Talk 14: Applications

Depending on interests of the speaker and participants, discuss some of the applications mentioned in [AHI24, Part II].

References

[AHI24] Toni Annala, Marc Hoyois, and Ryomei Iwasa. Atiyah duality for motivic spectra. *arXiv preprint arXiv:2403.01561*, 2024.

[AHI25] Toni Annala, Marc Hoyois, and Ryomei Iwasa. Algebraic cobordism and a conner–floyd isomorphism for algebraic k-theory. *Journal of the American Mathematical Society*, 38(1):243–289, 2025.

[AI22] Toni Annala and Ryomei Iwasa. Motivic spectra and universality of k-theory. *arXiv preprint arXiv:2204.03434*, 2022.