Arithmetic representations of fundamental groups

GAUS-AG SoSe 2025

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Introduction

In this GAUS AG we study the paper [Lit21]. Let k be a finitely generated field, let X be a smooth curve over k and let ℓ be a prime different from the characteristic of k. A finite-dimensional representation ρ of $\pi_1(X_{\overline{k}})$ over an ℓ-adic field L is called *arithmetic* if it appears as a subquotient of the restriction of a representation of $\pi_1(X_{k'})$ over L, where k'/k is a finite extension. The main example of (and motivation for studying) arithmetic representations are representations arising from geometry.

The author proves several finiteness results for arithmetic representations that are analogues of the Shafarevich, Fontaine–Mazur and Frey–Mazur conjectures. Namely, fix a representation $\overline{\rho}\colon \pi_1(X_{\overline{k}}) \to \mathrm{GL}_n(\mathbb{F}_{\ell^r})$. Then (Theorem 1.1.3 and Corollary 1.1.5) the set of isomorphism classes of semisimple arithmetic representations

$$\rho\colon \pi_1(X_{\overline{k}}) \to \mathrm{GL}_n(L) \quad \text{with} \quad \mathrm{Tr}(\rho) \equiv \mathrm{Tr}(\overline{\rho}) \mod \mathfrak{m}_L$$

is finite.[1]

For the proof, the author uses a theorem of Lafforgue to show that all semisimple arithmetic representations over $L = \mathbb{C}_\ell$ are already defined over $\overline{\mathbb{Q}}_\ell$. Moreover, specialisation techniques reduce the problem to the case of finite fields k. Here, one uses deformation theory to arrive at a contradiction: infinitely many arithmetic lifts would mean a positive dimensional rigid subspace in the rigid generic fibre of the space of pseudo-representations lifting $\overline{\rho}$, which would have $\mathbb{C}_\ell$-points that are not $\overline{\mathbb{Q}}_\ell$-points.

The aim of the seminar is to introduce background material on (pseudo-)representations and deformation theory as well as rigid geometry to follow the proof of Theorem 1.1.3. Towards the end of the GAUS AG, we turn to the more quantitative Theorem 1.1.10. Namely, the above finiteness result says that for any given semisimple arithmetic representation $\widetilde{\rho}\colon \pi_1(X_{\overline{k}}) \to \mathrm{GL}_n(\overline{\mathbb{Z}}_\ell)$ lifting $\overline{\rho}$ there exists a ball around $\widetilde{\rho}$ such that $\widetilde{\rho}$ is the only semisimple arithmetic lift of $\overline{\rho}$ in that ball. Theorem 1.1.10 gives an explicit bound for the radius of this ball. We will introduce the relevant weight filtrations on deformation rings and sketch the proof of Theorem 1.1.10.

[1] Restricting to tame representations, one can remove the residual condition, i.e. there are only finitely many semisimple arithmetic n-dimensional tame representations of $\pi_1(X_{\overline{k}})$ over L.

Time and Place

- We will meet in person according to the schedule detailed below on *Thursdays* in Frankfurt (Robert-Mayer Str. 6-8, Room 309) or Heidelberg (INF 230 (next to Mathematikon!), kleiner Hörsaal 001).
- A session will consist of two talks. Usually the talks are somewhere between 60 to 90 minutes long. An example session could look as follows:

#	Session
1.	14 : 00 - 15 : 15
☺	coffee break
2.	15 : 45 - 17 : 00

Schedule

Introduction and deformation spaces *(Heidelberg, 08.05.2025)*

Talk 0. Introduction (60 minutes). *(Speaker: Marius)*.
Define the class of arithmetic representations [Lit21, Definition 1.1.1]. Follow with Example 1.1.2 of [Lit21] and sketch the details. Then state the first finiteness results: [Lit21, Theorem 1.1.3, Corollary 1.1.5]. Mention the finiteness conjectures of Fontaine–Mazur: [FM95, Conjectures 2a and 2b]. Proceed by sketching the proof of Theorem 1.1.3, following Litt's presentation. Then state [Lit21, Theorem 1.1.10], and outline its proof, once again as in Litt. If time permits, briefly mention the finiteness results of Deligne–Esnault–Kerz [EK12] and Deligne [Del87], and discuss how they relate to Litt's finiteness results.
Talk 1. Deformation spaces (90-120 minutes). *(Speaker: Benjamin)*.
The goal of this talk is to introduce deformation rings. While we will not follow the presentation in Mazur's masterful article [Maz89], it is highly recommended to turn to for intuition and examples. Instead, we follow [Böc13, §1.1–1.5]. Begin by introducing Mazur's finiteness condition, then present Examples 1.2.2 from [Böc13]. Continue by stating and proving Proposition 1.3.1, but as Böckle does, defer the proof of the Noetherianness to later. If time permits, follow [Maz89, §1.4], and explain the structure of the deformation ring associated to a one-dimensional residual representation. Other examples which could be included are found in [Maz89, §1.9]. Proceed by proving Proposition 1.5.1 of [Böc13], referring back to section 1.4 as necessary. Finish with Corollary 1.5.2.

Arithmetic representations and determinants *(Frankfurt, 15.05.2025)*
Talk 2. Basics on arithmetic representations (60 minutes). *(Speaker: Nils)*.
The goal of this talk is to introduce some basics on arithmetic representations. Start by recalling the notion of an arithmetic representation, and prove Proposition 3.1.1 of [Lit21], and Corollary 3.1.3. Define the field of moduli and field of definition of a semisimple arithmetic representation, and prove that they are equal for number fields [Lit21, Theorem 3.1.5].
Talk 3. Determinants à la Chenevier (90 minutes). *(Speaker: Alireza)*.
Define pseudorepresenations [Lit21, Definition 2.1.1]. A good source for pseudorepresentations in the generality of Chenevier is [Eri13], but it contains (much) more than we will cover. When defining pseudorepresentations, it might be easier to first introduce [Böc13, Definition 2.2.2], which works well in characteristic zero, and then explain how Chenevier's definition works well in arbitrary characteristic. Prove that a representation gives rise to a pseudorepresentation, and state that conjugacy classes of semisimple representations over an algebraically closed field are in bijection with pseudorepresentations (see, for example [Eri13, Theorem 1.3.11]). Include a proof to the extent that it is possible - it is alright to prove some special case, and to assume standard results from representation theory. Strive to at least state what results from representation theory you are using.

Rigid analytic geometry and the character variety *(Heidelberg, 22.05.2025)*

Talk 4. Basics of rigid analytic geometry (90 minutes). *(Speaker: Immanuel).*
The purpose of this talk is to introduce the basics of rigid analytic geometry, enough to understand Chenevier's rigid analytic character variety. The speaker should look at Section 1 of [Che], and coordinate with the speaker of the next talk so that everything that they need is covered. Emphasise intuition and examples before proofs, and give the general structure of the theory. Other sources for rigid analytic geometry which can be useful are [Con08] and [Bos14].

Talk 5. Construction of the character variety (90 minutes). *(Speaker: ???).*
The goal of this talk is to sketch the construction of the character variety of Chenevier – a good overview is the presentation of [Che, §2]. It is alright to assume that $p > n$, as Chenevier does. Start by defining the moduli problem, and define the p-adic character variety as the rigid analytic space representing the moduli problem. Then prove Theorem 2.2. It is not necessary to prove Lemma 2.3 in detail, but sketch the important parts. For the proof of Theorem 2.1 and 2.2, you should state the Nyssen–Rouquier–Procesi's result, but do not prove it. Finish by proving Proposition 2.4.

Rigidity and dynamics *(Frankfurt, 12.06.2025)*

Talk 6. Rigidity of arithmetic representations (90 minutes). *(Speaker: ???).*
Start by proving Lemma 3.2.1 of [Lit21]. Recall the notion of weights from Deligne's Weil II, and how Lafforgue's result are applied, but do not meditate on automorphic theory. If time allows, sketch the proof of Carayol's result. Then proceed by proving Theorem 3.2.3. This should be relatively straightforward, given the work in the previous lecture. Chenevier's Proposition G does not have to be proven, but should be mentioned.

Talk 7. Dynamical Mordell Lang (90 minutes). *(Speaker: Leonie).*
The purpose of this talk is to prove Corollary 4.1.6 of [Lit21]. Proceed simply by following the presentation of [Lit21], Section 4.1. State, but do not prove the l-adic analytic arc lemma of Poonen. Present both Lemma 4.1.3, and Corollary 4.1.5, and include their proofs.

Back-up slot *(Heidelberg, 26.06.2025)*

Finiteness and weight filtrations *(Frankfurt, 17.07.2025)*

Talk 8. Finiteness of arithmetic representations (90 minutes). *(Speaker: Jakob).*
Following Section 4.2 of [Lit21], prove Theorem 1.1.3 and 1.1.5. In the proof of 1.1.3, state the Weierstress preparation theorem but do not prove it, and state the results regarding specialization as well. Blackbox the theorem of Deligne in the proof of Theorem 1.1.5.

Talk 9. Weight filtrations on deformation rings I (90 minutes). *(Speaker: Ruth).*
First introduce the weight-filtration on the first cohomology of an arithmetic representation. Explain the relation to the ordinary weight filtration for representations of geometric origin and weight zero. Move on to defining the weight filtration on the deformation ring [Lit21, Definition 5.1.3]. Then prove Lemma 5.1.5: start by proving the number field case, and only

include Serre's argument in case there is time. Briefly explain the used results of Liu–Zhu and Diao–Lan–Liu–Zhu, but do not spend much time on it.

Weight filtrations and Frey–Mazur *(Heidelberg, 24.07.2024)*

Talk 10. Weight filtration on deformation rings II (90 minutes). *(Speaker: ???)*.
Define the index of homothety as in [Lit21, Definition 5.1.7]. Then proceed by proving the analogue of Lemma 5.1.5 for the deformation ring. The proof of 5.1.8 is quite long. It would be preferable to present Lemma 5.1.10 in full, but if there is a lack of time, skip (2), (3) and (4) of Lemma 5.1.10.

Talk 11. Integral analysis and proof of Frey–Mazur (90 minutes). *(Speaker: ???)*. Prove Theorem 1.1.10. You do not have to prove the results of Section 5.2, but intuition for why one can take the constants as claimed is desirable. Further, it would be helpful to state Lemma 5.2.2, and give some heuristic argument for why it holds. The inequality (5.3.1) does not have to be rigorously proven.

Talk ☺. Dinner / hike in Heidelberg.

References

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