Categorical Deligne–Langlands correspondence

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The categorical Deligne–Langlands correspondence can be viewed as a special case of Fargues–Scholze's geometrization conjecture for the principal block of smooth representations of a p-adic group. In the seminar, we mostly study Hellmann's paper on the subject [Hel23]. The final talks discuss the progress by Ben Zvi, Chen, Helm and Nadler [Ben+24], focussing on results for the general linear group.

Comments

The program assumes familiarity with algebraic geometry and reductive groups (for a quick and dirty reference about reductive groups see [Voe], for more substantial introductions [Con] or [S M]). The main part of the seminar is based on Hellmann's paper, supplemented by additional literature where appropriate. For the final talks, a working knowledge of ∞-categories is expected. Completeness is not the goal. Nonetheless, every participant should be able to gain insight from each talk.

The talks marked with an asterisk $(*)$ are independent of the rest of the seminar and are suitable for advanced Master students. These are **Talks 1, 2, 3 and 5**.

A central aim of the seminar is to develop an intuition for the style and structure of arguments in this area. As a general guideline, each talk should include a detailed presentation of at least one representative argument. Speakers are free to choose which argument to highlight and are encouraged to consult with other participants, ideally ahead of their talk.

If you are interested in participating but uncertain about which talk to choose, please do not hesitate to contact the organizers.

Time and place

The seminar takes place in a hybrid format.

- Tuesdays, 14:00 – 15:30, including a short break.
- Start date: April 22, end date: July 15
- The seminar will take place in Darmstadt, Room S215 401.
- Zoom meeting ID: 612 2072 7363, Password: Largest six digit prime number.

Talks

Leitfaden (Apr 22)

This is an overview talk (20–30 minutes) given by one of the organizers explaining the interrelation of talks and the structure of the seminar.

Talk 1* - Weil–Deligne representations (Apr 22, only 60 minutes)

The references are [You, pp. 1.1, 1.2], [Com] and [BH06, §§31–32]. Recall the structure of the absolute Galois group of a non-Archimedean local field F, i.e., a finite extension of $\mathbb{F}_p((t))$ or $\mathbb{Q}_p$ for some prime number p, as necessary to introduce the Weil group W_F. Explain the equivalence between finite dimensional, continuous representations of W_F and Weil–Deligne representations [Com, Theorem 3.5] (see also [BH06, §32.6]).

Talk 2* - Spaces of L-parameters (Apr 29)

The reference is [Hel23, §2.1]. For a linear algebraic group G over a field of characteristic zero, introduce the spaces of L-parameters X_G, and study their basic geometry. In particular define the regular locus. Focus on the case $G = \mathrm{GL}_n$. Keep the discussion of derived schemes/stacks after Corollary 2.11 short. These are formally needed in Talks 3 & 4 but we will not work with them explicitly.

Talk 3* - Derived categories of quasi-coherent sheaves (May 06)

Follow [Hel23, §2.2]. Introduce what we need about the categories of quasi-coherent sheaves on stacks [Alp, Section 6.1] and their derived versions [Ols07, Section 4]. Briefly mention that the theory extends to derived stacks [GR17]. Keep in mind that we formally these but do not work with them explicitly for most of the seminar. Explain why the use of derived stacks, e.g., in Lemma 2.16 is necessary, see the discussion at the end of Section 2.2.

Talk 4 - Self-duality of derived push forwards (May 13)

The reference is [Hel23, §2.3]. Discuss Conjecture 2.17 and its proof for $G = \mathrm{GL}_2, \mathrm{GL}_3$ in Proposition 2.21. For notions about duality and dualizing complexes (for schemes) see [Aut, 0DWE] or [GW23, Section (25.14)ff]. See also [Lur] for a quick proof of the Borel–Weil–Bott theorem.

Talk 5* - Smooth representations of p-adic groups (May 20)

Give an introduction to smooth representations of p-adic groups. The reference is [BH06, §§1–2] (see also [Ber] and [Zhu]). Introduce locally profinite groups (§1.1), smooth representations (§2.1), and (compact) induction and Frobenius reciprocity (§§2.4–2.5). Keep these topics brief. Explain the equivalence between smooth representations and smooth (or, non-degenerate) modules under the Hecke algebra (§4.2) (see also [Ber, §2.1] and [Zhu, §7]). Focus on GL_n, where also [Wed00] might be helpful. If time permits, explain how induction/restriction translates under this equivalence [Hel23, Equations (3.5), (3.6)].

Talk 6 - Bernstein blocks (May 27)

The aim is to give details on the Bernstein decomposition and the Bernstein isomorphism as in the first and second displayed formula in [Hel23, §3.1]. References are [Ber84] (see also [Ber, pp. III, 2.2]) and [Hai14, §3], restricting to the case of split groups.

Talk 7 - Formulation of the main conjecture (Jun 03)

The reference is [Hel23, §§3.1–3.2]. Follow the discussion in §3.1 and formulate Conjecture 3.2. In particular, explain the relation to local class field theory.

Talk 8 - The Kazhdan–Lusztig classification (Jun 10)

State the Kazhdan–Lusztig classification [KL87, Theorem 7.12]. Give with some details as time permits, focussing on $G = \mathrm{GL}_n$ when suitable. Explain the relation to Hellmann's conjecture [Hel23, Remark 3.3(b)].

Talk 9 - Modified local Langlands for the principal block of GL_n (Jun 17)

The reference is [Hel23, §4.1]. Explain the construction of the modified local Langlands correspondence for the principal block of GL_n, and make the relation to the Kazhdan–Lusztig classification as time permits (Remark 4.4).

Talk 10 - Interpolation in families (Jun 24)

The reference is [Hel23, §4.2]. Explain the construction of the Emerton–Helm family $\tilde{\mathcal{V}}_G$, conjecturally interpolating the modified local Langlands correspondence (Conjecture 4.8).

Talk 11 - Main conjecture in the regular case (Jul 01)

The reference is [Hel23, §§4.4–4.5]. Sketch the proof of the main conjecture for GL_n over the regular locus of $X_{\check{G}}$ (Theorem 4.25). Point out that Conjecture 4.14 implies the main conjecture for GL_n in general.

Talk 12 - Proof of the main conjecture for GL_2 (Jul 08)

The reference is [Hel23, §4.8]. Prove the main conjecture for GL_2 from the regular case together with a proof of Conjecture 4.14 for GL_2 (Proposition 4.29).

Talks 13 & 14 (Jul 15, double session and dinner)

The goal is to present an overview of the proof of [Ben+24, Theorem 1.9]. Speakers are expected to coordinate among themselves regarding the division of the material. Note that in the paper, the roles of the group G and its dual $\check{G}$ are interchanged.

References

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