# $\mathscr{D}$-elliptic sheaves and the Hasse principle

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The seminar will take place on Fridays, 9-11, in room SR8.

The goal of the seminar is to study the recent paper by Arai-Hattori-Kondo-Papikian [Ara+24]. We will have some preliminary talks on stacks and $\mathscr{D}$-elliptic sheaves.

## 1. Introduction

Let $p$ be a prime number and $q$ a power of $p$. Let $A = \mathbb{F}_q[t]$ be the polynomial ring in a variable $t$ and $F$ its fraction field. Let $\infty$ denote the place of $F$ corresponding to $1/t$. Moreover, let $d \geq 2$ be an integer and $D$ a central division algebra over $F$ of dimension $d^2$, split at $\infty$ and such that, at any place $x$ at which $D$ does not split, $\mathrm{inv}_x(D) = \frac{1}{d}$.
A *$\mathscr{D}$-elliptic sheaf* on an $\mathbb{F}_q$-scheme $S$ is a vector bundle of rank $d^2$ on $\mathbb{P}^1_{\mathbb{F}_q} \times_{\mathbb{F}_q} S$, equipped with an action of a sheafified version $\mathscr{D}$ of $D$, together with a meromorphic $\mathscr{D}$-linear Frobenius with a pole at $\infty$, satisfying some periodicity conditions. These objects were first considered in the seminal work of Laumon-Rapoport-Stuhler [LRS93], generalizing earlier work by Drinfeld on elliptic sheaves, for the sake of proving the local Langlands correspondence over function fields in higher dimensions. It turns out that the moduli stack classifying $\mathscr{D}$-elliptic sheaves on $F$-schemes is a Deligne-Mumford stack and consequently it admits a coarse moduli space, a scheme denoted by $X^D$ and called the *Drinfeld-Stuhler variety.* In the dictionary of the Langlands correspondence, $\mathscr{D}$-elliptic sheaves are the function field analogues of polarized abelian varieties equipped with an action of an indefinite quaternion algebra $B$ over $\mathbb{Q}$. Therefore, for instance when $d = 2$, we expect $X^D$ to mirror the properties of the Shimura curve $V_B$ attached to $B$.
In this vein, in [Ara+24] the authors prove an obstruction to the Hasse principle over some quadratic extensions of $F$, generalizing a similar result for $V_B$ in [Jor86]. More precisely, in [Jor86], Jordan gives an example of an indefinite quaternion algebra $B/\mathbb{Q}$ such that $V_B$ has no rational points over a certain quadratic extension $E$ but admits $E_v$ points for all places $v$ of $E$. In this seminar, we aim to study the construction of the aforementioned obstruction in [Ara+24]. In the process, we will also cover the basics of stacks, $\mathscr{D}$-elliptic sheaves and $t$-motives.

## 2. Talks

### 0. **Overview** (Date: 25.04.25, Speaker: *Sriram C V*)

Begin by recalling Drinfeld's theory of elliptic sheaves (or Shtukas) ([**Mumford**; Dri87]), and explain briefly the importance of considering generalizations of them equipped with an action of $\mathscr{D}$, namely $\mathscr{D}$-elliptic sheaves. Then motivate and explain the main result in [Ara+24], by summarizing each talk of the seminar.

### 1. **Stacks I** (Date: 02.05.25, Speaker: )

Give a quick overview of 2-categories and present the $(2, 1)$-category of groupoids as an example. Define categories fibred in groupoids and explain why they are essentially the same as functors with values in the $(2, 1)$-category of groupoids (see Stacks 003S or [Beh+07, p. 37] after the exercise). Discuss examples 2.1, 2.3, 2.6 and 2.7 from [Beh+07]. Briefly recall faithfully flat descent and Grothendieck topologies[1]. Define stacks (you can for instance follow [Gvi15, §4] or [Beh+07, §4.1-2]). Based on time, choose all or some of the four examples above and explain why they are stacks.

### 2. $\mathscr{D}$-elliptic sheaves (Date: 09.05.25, Speaker: )

We aim to cover [Ara+24, §2] in this talk. An alternate reference for this talk is also [BS97, §4.4]. Begin by briefly recalling, without proofs, the theory of central simple algebras required here, for instance Hasse invariants etc. Then present the definition of a $\mathcal{D}$-elliptic sheaf (assume the definition of $p$-divisible group from [Ara+24, §5] when explaining [Ara+24, Def. 2.1(6)]), level $I$-structures and the moduli stack of $\mathscr{D}$-elliptic sheaves with level $I$-structure, $\mathcal{E}ll_{\mathscr{D},I}$. Define the $t$-motive associated to a $\mathcal{D}$-elliptic sheaf and discuss [Ara+24, §2.3]. For more details on $t$-motives, one could also look at [And86].

### 3. Stacks II (Date: 16.05.25, Speaker: )

The goal of this talk is to discuss Deligne-Mumford stacks and then prove that $\mathcal{E}\ell\ell_{\mathscr{D},I}$ is a smooth Deligne-Mumford stack ([LRS93, Theorem 4.1]). For the first part of the talk, you can for instance follow [Beh+07, §5.1-2] or [Gvi15, §5]. Smoothness and properness of a D-M stack should be covered. For the second part, explain the ideas of the proof of [LRS93, Theorem 4.1] without giving all the details. At the end, mention [LRS93, Theorem 5.1] and [LRS93, Corollary 6.2] without proofs.

### 4. The Drinfeld-Stuhler variety (Date: 23.05.25, Speaker: )

Recall the stack $\mathcal{E}\ell\ell_{\mathscr{D},I}$ from previous talk, define the Drinfeld-Stuhler variety, $X^D$, as the coarse moduli space of $\mathcal{E}\ell\ell_{\mathscr{D},\emptyset|F}$, and discuss the relevant details in [Ara+24, §3.3] (before Definition 3.5) describing $X^D$. Finally, prove [Ara+24, Thm.3.8] regarding the $K$-points of $X^D$, for a finite extension $K$ of $F$, after proving the results from [Ara+24, §3.1-2].

### 5. $\mathscr{D}$-elliptic sheaves over finite fields (Date: 30.05.25, Speaker: )

For a $\mathscr{D}$-elliptic sheaf $\underline{\mathcal{E}}$, define $\mathrm{End}(\underline{\mathcal{E}})$ as well as $\mathrm{Aut}(\underline{\mathcal{E}})$. Over a finite field, one can show that, for a sound $\underline{\mathcal{E}}$, $\mathrm{Aut}(\underline{\mathcal{E}})$ is a cyclic group of order dividing $q^d - 1$. Discuss the proof of this [Ara+24, Prop.4.3]. Then discuss the potential good reduction of $\mathscr{D}$-elliptic sheaves [Ara+24, Lemma 4.14] and bound the degree of field extension admitting good reduction [Ara+24, Prop. 4.16]. Depending on time, also discuss the determination of the center of $F \otimes_A \mathrm{End}(\underline{\mathcal{E}})$ [Ara+24, §4.2] as well as the structure of the endomorphism algebra [Ara+24, §4.3].

### 6. $\mathfrak{p}$-adic properties (Date: 27.06.25, Speaker: *Alireza Shavali*)

Begin by defining a $\varphi$-sheaf over a local $\mathbb{F}_q$-algebra and giving the associated construction Gr(-) for such objects together with its properties [Dri87, Prop. 2.1]. Then recall the notion of a $\mathfrak{p}$-divisible group from [Tag93, §1.2], for $\mathfrak{p} \in |X| \setminus \infty$. Using these, define the Tate module $T_{\mathfrak{p}}(\underline{\mathcal{E}})$ of a $\mathscr{D}$-elliptic sheaf $\underline{\mathcal{E}}$ and discuss its properties as well as the reduced characteristic polynomial of Frobenius [Ara+24, §5.3, §5.4]. Conclude the talk by bounding the image of the inertia groups [Ara+24, §5.5].

### 7. Determinants of $\mathscr{D}$-elliptic sheaves (Date: 04.07.25, Speaker: )

Define $(\mathbb{F}_p, \varphi)$-sheaves [Ara+24, Definition 6.1] and their determinant. Consider the $t$-motive $P$ associated with a sound $\mathscr{D}$-elliptic sheaf $\underline{\mathcal{E}}$ over $L$, and attach to it a determinant $Q$. Explain the relation of the determinant with the $\mathfrak{p}$-torsion of $\underline{\mathcal{E}}$ [Ara+24, Proposition 6.5].

[1] This can be omitted if time does not allow.

Finally, consider the Carlitz module $C$, and define the associated mod $\mathfrak{p}$ Galois character $\chi_{C,\mathfrak{p}}$. Using the results just proved, prove the relation between the restrictions to inertia of $\chi_{C,\mathfrak{p}}$ and the character $\delta_{\underline{\mathcal{E}},\mathfrak{p}}$ attached to a $\mathscr{D}$-elliptic sheaf $\mathcal{E}$ in the beginning of Section 6.

### 8. The canonical isogeny character (Date: 11.07.25, Speaker: )

For a $\mathscr{D}$-elliptic sheaf $\underline{\mathcal{E}}$, define the *canonical isogeny character* $\rho_{\underline{\mathcal{E}},\mathfrak{p}} : G_L \to \mathbb{F}^\times$, and the representation $\pi_{\underline{\mathcal{E}},\mathfrak{p}} : G_L \to \mathrm{Aut}_{\mathbb{F}}(\underline{\mathcal{E}}[\mathfrak{p}](L^{\mathrm{sep}})$. Relate the determinant of $\pi_{\underline{\mathcal{E}},\mathfrak{p}}$ with $\rho_{\underline{\mathcal{E}},\mathfrak{p}}$ as in [Ara+24, Lemma 7.1], and with $\delta_{\underline{\mathcal{E}},\mathfrak{p}}$ as in Lemma 7.2. Describe the mod $\mathfrak{p}$ characteristic polynomial of Frobenius as in Lemma 7.3.

Prove Proposition 7.5 for the case when $\underline{\mathcal{E}}$ is a sound $\mathscr{D}$-elliptic sheaf over a finite extension of $F_\infty$, of generic characteristic and good reduction. Deduce Corollary 7.6 about the local Galois representation for a global $\mathscr{D}$.

Compose the canonical isogeny character with a reciprocity map at a place $v$ to obtain the character $\widetilde{r}_{\underline{\mathcal{E}},\mathfrak{p}}$ of Section 7.4, and deduce its properties from the results on the Galois side.

### 9. Global points and the Hasse principle (Date: 18.07.25, Speaker: )

Describe the mod $\mathfrak{p}$ reduction of the canonical isogeny character [Ara+24, Proposition 8.1]. Prove the criterion Theorem 8.5 for the non-existence of global points on the Drinfeld-Stühler variety $X^D$. Give the Examples 8.6-8.7 (optionally, explain how they are computed in PARI/GP). If time allows, present the counterexample from the Hasse principle [Ara+24, Theorem 9.11], assuming the results of Papikian on the existence of local points, given in Lemmas 9.1, 9.3, 9.5, and deducing from them (or just stating) Lemma 9.7 and Proposition 9.9 from them.

## References


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