Blockseminar on $\mathrm{A_{inf}}$-cohomology

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Program by Lucas Gerth

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Abstract

The goal of this workshop is to provide an introduction to the $\mathrm{A_{inf}}$-cohomology, defined by Bhatt–Morrow–Scholze in [BMS18].

In what follows we fix a prime number p. Let K be a finite extension of $\mathbb{Q}_p$ with rings of integers $\mathcal{O}_K$ and residue field k. We fix an algebraic closure $\overline{K}$ of K and denote by C its completion. Let X be a smooth proper algebraic variety over K with good reduction, i.e $X = \mathfrak{X}_K$ for $\mathfrak{X}$ a proper smooth scheme over $\mathcal{O}_K$. More generally, we allow X to be a rigid space and $\mathfrak{X}$ to be a formal scheme over $\mathcal{O}_K$ with adic generic fiber X. One can consider the p-adic étale cohomology $H^n_{\text{ét}}(X_C, \mathbb{Q}_p)$, a finite-dimensional linear representation of the absolute Galois group $G_K = \mathrm{Gal}(\overline{K}/K)$. Using the good reduction hypothesis, there is a second cohomology theory available, the crystalline cohomology $H^n_{\mathrm{crys}}(\mathfrak{X}_k\,/W(k))$. This consists of a finite $W(k)$-module with a Frobenius-semilinear endomorphism φ that becomes an isomorphism after inverting p. Grothendieck suggested the existence of a "mysterious functor" relating these two p-adic cohomology theories. This was clarified by Fontaine, who introduced his *crystalline period ring* B_{crys}, a topological $K_0 = W(k)[\frac{1}{p}]$-algebra with a semilinear action of Gal_K and a Frobenius endomorphism φ, satisfying

$$\mathrm{B}_{\mathrm{crys}}^{\mathrm{G_K}} = K_0\,, \quad \mathrm{B}_{\mathrm{crys}}^{\varphi=1} \cap \mathrm{B}_{\mathrm{dR}}^{+} = \mathbb{Q}_p\,.$$

Fontaine then formulates his C_{crys}-conjecture, which was proven in this generality by Colmez–Niziol [CN15].

Conjecture 1. (C_{crys}) There exists a natural isomorphism, compatible with Galois action and Frobenius

$$H^n_{\text{ét}}(X_C, \mathbb{Q}_p) \otimes_{\mathbb{Q}_p} B_{\mathrm{crys}} \cong H^n_{\mathrm{crys}}(\mathfrak{X}_k\,/W(k)) \otimes_{W(k)} B_{\mathrm{crys}}. \tag{1}$$

In particular, one can recover the rational crystalline cohomology with its Frobenius from the étale cohomology via the formula

$$H^n_{\mathrm{crys}}(\mathfrak{X}_k\,/W(k))[\tfrac{1}{p}] = (H^n_{\text{ét}}(X_C, \mathbb{Q}_p) \otimes_{\mathbb{Q}_p} B_{\mathrm{crys}})^{\mathrm{Gal}_K} \tag{2}$$

On the other hand, Fontaine also showed that one can recover the p-adic étale cohomology from the crystalline cohomology as follow

$$H^n_{\text{ét}}(X_C, \mathbb{Q}_p) = (H^n_{\mathrm{crys}}(\mathfrak{X}_k\,/W(k))[\tfrac{1}{p}] \otimes_{K_0} \mathrm{B}_{\mathrm{crys}})^{\varphi=1} \cap \mathrm{Fil}^0(H^n_{\mathrm{crys}}(\mathfrak{X}_k\,/W(k))[\tfrac{1}{p}] \otimes_{K_0} \mathrm{B}_{\mathrm{dR}}), \tag{3}$$

where for the last term on the right-hand side, we use the isomorphism $H^n_{\mathrm{crys}}(\mathfrak{X}_k\,/W(k))[\frac{1}{p}] \otimes_{K_0} K \cong H^n_{\mathrm{dR}}(X/K)$ and we tensor the Hodge filtration with the natural filtration $\xi^i\,\mathrm{B}_{\mathrm{dR}}^{+}$ on B_{dR}.

There are two caveats around the crystalline comparison Theorem (1).

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1. Firstly, the p-adic étale cohomology also has an integral structure $H^n_{\text{ét}}(X_C, \mathbb{Z}_p)$ and it is natural to ask for integral analogs of the formulas (2) and (3). Unfortunately, this does not hold in general, due to the presence of torsion on either side (among other things). There are still positive results when the degree n and the ramification index e of $K/\mathbb{Q}_p$ are not too big, i.e. when $ne < p-1$, roughly using the integral variant A_{crys} of B_{crys}, see e.g. [Car08].

2. Second, if X is instead assumed to be a smooth proper rigid space directly defined over C, with integral model $\mathfrak{X}$ over $\mathcal{O}_C$, one would like an analog of the formula (1) involving $H^n_{\text{ét}}(X, \mathbb{Q}_p)$ on the left-hand side and the crystalline cohomology $H^n_{\text{crys}}(\mathfrak{X}_{\overline{k}}/W(\overline{k}))$ on the right-hand side[1].

In [BMS18], the authors solve both problems by constructing a cohomology theory taking values in mixed characteristic analogs of Dieudonné modules. More precisely:

Theorem 2. *([BMS18, Thm. 1.8]) Let $\mathfrak{X}$ be a proper smooth formal scheme over $\mathcal{O}_C$, where C is* any *complete algebraically closed extension of $\mathbb{Q}_p$ with residue field k. There is a cohomology theory*

$$\mathrm{R}\Gamma_{\mathrm{A}_{\inf}}(\mathfrak{X})$$

with values in perfect complexes of Breuil–Kisin–Fargues modules, that is, a perfect complex of $\mathrm{A}_{\inf} = W(\mathcal{O}_{C^\flat})$-modules equipped with a Frobenius-semilinear map $\varphi\colon \mathrm{R}\Gamma_{\mathrm{A}_{\inf}}(\mathfrak{X}) \to \mathrm{R}\Gamma_{\mathrm{A}_{\inf}}(\mathfrak{X})$ that becomes a quasi-isomorphism after inverting ξ. It satisfies the following compatibilities

1. *With crystalline cohomology:*

$$\mathrm{R}\Gamma_{\mathrm{A}_{\inf}}(\mathfrak{X}) \otimes^{\mathbb{L}}_{\mathrm{A}_{\inf}} W(k) \simeq \mathrm{R}\Gamma_{\text{crys}}(\mathfrak{X}_k/W(k)).$$

More generally,

$$\mathrm{R}\Gamma_{\mathrm{A}_{\inf}}(\mathfrak{X}) \otimes^{\mathbb{L}}_{\mathrm{A}_{\inf}} \mathrm{A}_{\text{crys}} \simeq \mathrm{R}\Gamma_{\text{crys}}(\mathfrak{X}_{\mathcal{O}_C/p}/\mathrm{A}_{\text{crys}}).$$

2. *With de Rham cohomology:*

$$\mathrm{R}\Gamma_{\mathrm{A}_{\inf}}(\mathfrak{X}) \otimes^{\mathbb{L}}_{\mathrm{A}_{\inf}} \mathcal{O}_C \simeq \mathrm{R}\Gamma_{\mathrm{dR}}(\mathfrak{X}/\mathcal{O}_C).$$

3. *With étale cohomology:*

$$\mathrm{R}\Gamma_{\mathrm{A}_{\inf}}(\mathfrak{X}) \otimes_{\mathrm{A}_{\inf}} \mathrm{A}_{\inf}[\tfrac{1}{\mu}] \simeq \mathrm{R}\Gamma_{\text{ét}}(X, \mathbb{Z}_p) \otimes_{\mathbb{Z}_p} \mathrm{A}_{\inf}[\tfrac{1}{\mu}].$$

One can derive from Theorem 2 a new proof of the C_{crys}-conjecture that applies to smooth proper formal schemes defined over $\mathcal{O}_C$.

We see that $\mathrm{R}\Gamma_{\mathrm{A}_{\inf}}(\mathfrak{X})$ offers a refinement of various p-adic cohomology theories, whose coefficients form subsets of $\mathrm{Spec}(\mathrm{A}_{\inf})$, that admit comparison theorems whenever these subsets overlap. However, $\mathrm{R}\Gamma_{\mathrm{A}_{\inf}}(\mathfrak{X})$ is a finer invariant that cannot be recovered from other known cohomology theories. The Breuil–Kisin–Fargues modules appearing above are a generalization of the notion of Breuil–Kisin modules [Kis06][Bre00]. The latter are defined, for K discretely valued, as certain $\mathfrak{S} = W(k)[[u]]$-modules with a Frobenius. Using the theory of Breuil–Kisin (resp. Breuil–Kisin–Fargues) modules, the authors also prove the following.

Corollary 3. *([BMS18, Thm. 1.1.(iii), Thm. 1.4]) Let $\mathfrak{X}$ be a proper smooth formal scheme over $\mathcal{O}_K$ or over $\mathcal{O}_C$.*

1. *If $H^n_{\text{crys}}(\mathfrak{X}_k/W(k))$ is torsion-free, so is $H^n_{\text{ét}}(X_C, \mathbb{Z}_p)$.*

2. *Assume that $H^n_{\text{crys}}(\mathfrak{X}_k/W(k))$ and $H^{n+1}_{\text{crys}}(\mathfrak{X}_k/W(k))$ are torsion-free. Then $H^n_{\text{crys}}(\mathfrak{X}_k/W(k))$ can be recovered functorially from $H^n_{\text{ét}}(X_C, \mathbb{Z}_p)$.*

[1] It will become apparent as we learn more about the paper that, rather than the special fiber $\mathfrak{X}_{\overline{k}}$, it is more advantageous to remember $\mathfrak{X}_{\mathcal{O}_C/p}$.

Informations

We will follow the original paper [BMS18]. As the paper is long and very technical, we will have to omit some parts. We will mainly focus on the construction of the $\mathrm{A}_{\mathrm{inf}}$-cohomology, with a special emphasis on details. In particular, we won't prove the crystalline comparison theorem (§12) and we'll only sketch the comparison with the de Rham–Witt complex (§11).

Each talk will be one hour long. We will assume familiarity with algebraic geometry and to some extent non-archimedean geometry. The talks marked with a star $\star$ are very technical and the speaker should feel extra motivated. If you are interested in participating, feel free to contact one of the organizers.

Time and place:

This blockseminar will take place over the course of two days, Friday February 7 and Friday February 14, in Frankfurt, room 309, Robert-Mayer Strasse 6-8, 60325 Frankfurt am Main. Each day session will last from 9:30 AM to 3:15 PM.

0 Leitfaden

February 7, 9:30 - 9:45

An overview talk (15 minutes) given by one of the organizers, presenting the main goals of the workshop and describing the talks.

1 Preliminaries

February 7, 9:45 - 10:30 (only 45 minutes due to the Leitfaden)

Recall the notion of derived-completeness ([BMS18, §6.2] or [BS22, p. 1.2]). Briefly discuss integral perfectoid rings (§3). Define the maps $\theta_r, \widetilde{\theta}_r$ and the generators of their respective kernels $\xi_r, \widetilde{\xi}_r$. Mention the relation to perfectoid Huber pairs (Lemma 3.20-21). Following §5, recall the pro-étale site of a rigid space and define the sheaf version of θ_r and $\widetilde{\theta}_r$. Prove Theorem 5.7.

2 Breuil–Kisin–Fargues modules

February 7, 10:45 - 11:45

Following §4, explain some properties of vector bundles on distinguished open subsets of $\mathrm{Spec}(\mathrm{A}_{\mathrm{inf}})$. It might be a good idea to recall the picture of $\mathrm{Spec}(\mathrm{A}_{\mathrm{inf}})$ in [SW20, Fig. 12.1]. Sketch a proof of Lemma 4.18 and Corollary 4.20 which will be used in the very last talk. Define Breuil–Kisin–Fargues modules (§4.3) and discuss the twist $A_{\mathrm{inf}}\{1\}$. Prove Lemma 4.26 and state Fargues' Theorem 4.28.

3 The operator $L\eta$

February 7, 13:00 - 14:00

Following §6, define the Berthelot–Ogus operator $L\eta$ and explain some of its properties. Prove at least Lemma 6.8, as well as the key Lemma 8.11 and Proposition 6.12, whose importance should be stressed. Conclude by treating in detail the example of the q-de Rham complex (Ex. 7.7).

4 The complex $\widetilde{\Omega}_{\mathfrak{X}}$

February 7, 13:45 - 14:45

Following §8, define the complex $\widetilde{\Omega}_{\mathfrak{X}}$ and prove Theorem 8.3 and Theorem 8.15.

5 *The complex $A\Omega_{\mathfrak{X}}$

February 14, 9:30 - 10:30 *Ruth Wild*

Following §9, define the complex $A\Omega_{\mathfrak{X}}$, state Theorem 9.2 and prove points $(i), (ii)$. You may want to blackbox some of the technical results leading to Lemma 9.7.

6 *Relation to de Rham–Witt complexes

February 14, 10:45 - 11:45 *Amine Koubaa*

Recall some properties of the de Rham–Witt complex and its relation to crystalline cohomology (eg. the introduction of [LZ04]). Describe it in the case of a torus (§10.4). Following §11, explain how to build an F-V procomplex out of $R\Gamma_{\text{proét}}(X, \mathbb{A}_{\text{inf}})$. Prove the remaining point (iii) of Theorem 9.2, at least cover the case of a torus (§11.2). The speaker is encouraged to focus on providing intuition and ideas of the constructions, without giving all technical details.

7 *The $\mathrm{B}_{\mathrm{dR}}^+$-cohomology

February 14, 13:00 - 14:00 *Thiago Solovera e Nery*

Following §13, define the $\mathrm{B}_{\mathrm{dR}}^+$-cohomology (Def. 13.14 and 13.18). State without proof Conrad–Gabber's spreading (Corollary 13.16). Prove Theorem 13.19 and Theorem 13.1. We suggest the speaker looks at the paper [Guo21], specifically Theorem 1.2.7, to complement or replace parts of §13.

8 The $\mathrm{A}_{\mathrm{inf}}$-cohomology and the main Theorems

February 14, 14:15 - 15:15

State Theorem 14.1 and prove parts $(i), (ii), (iv)$. Define the $\mathrm{A}_{\mathrm{inf}}$-cohomology, state Theorem 14.3 and prove parts $(i), (ii), (iv)$. Finally state Theorem 14.5 and prove $(i), (iii)$.

References

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