Moduli of Quiver Representations and GIT Quotients

GAUS AG

Wintersemester 2024/25

Time: Thursday at 2.00 pm
Location: Robert-Mayer-Str. 10, Seminarraum 711 (small)
Format: Each meeting consists of two 60 minute talks with a 30 minute coffee & cookies break in between.
zoom: Meeting-ID: 690 9713 2594
Passcode: 662140
zoom link

Schedule

Date	Talks	Speakers
07. November	1.1 and 1.2	Kevin Kühn, Yiu-Man Wong
14. November	2.1 and 2.2	Nicole Müller, Felix Göbler
28. November	CRC-Colloquium	
05. December	Workshop	
12. December	3.1 and 3.2	Miguel Prado, Jeonghoon So
19. December	4.1 and 4.2	Arne Kuhrs, Johannes Horn
23. January	5.1 and 5.2	Martin Ulirsch, Andrei Bud
06. February	6.1 and 6.2	Andreas Gross, Pedro Souza
13. February	7.1 and 7.2	Rızacan Çiloğlu, Martin Möller

Talks

1.1 Introduction to Moduli

The talk provides the foundation for understanding how moduli problems are formalized in algebraic geometry and set the stage for further discussions on Geometric Invariant Theory (GIT) and moduli of quiver representations. The talk shall introduce moduli functors, fine moduli spaces and coarse moduli spaces, and illustrates theses concepts with examples. If time allows, one can comment on the more modern approach via moduli stacks. References are [Soi21, Section 2] and [Hos15, Section 2].

1.2 Algebraic Group Actions and Quotients

Algebraic group actions and the construction of quotient spaces in algebraic geometry will be introduced. Additionally, we will study basic properties of orbits, stabilizers, and fixed points. Furthermore, the concept of categorical and geometric quotients shall be discussed. The talk shall cover [Soi21, Section 6]. A more detailed reference is [Hos15, Section 3].

2.1 Affine GIT

In the context of affine varieties, we study Geometric Invariant Theory (GIT) with a focus on the construction of affine GIT quotients under the action of a reductive group. Attention will also be given to (linearly) reductive groups, examining their properties and their actions on affine schemes. Nagata's Theorem will be discussed, and if time permits, one can also comment on the proof. The talk will cover [Soi21, Section 7.1], with [Hos15, Section 4] serving as a more detailed reference.

2.2 Projective GIT

The goal of this talk is to extend the affine GIT developed in the previous talk to the projective setting. The idea is to construct projective GIT quotients by gluing affine quotients. However, challenges arise when it is not possible to find open invariant affine subsets. To tackle this issue, we will concentrate on linearizations, an essential concept for navigating the challenges in the projective setting. We will define (semi-)stability and present the Hilbert–Mumford criterion for verifying it. The main reference will be [Hos15, Section 5], though we will also consult [Soi21, Sections 7.2–7.3] for an overview.

3.1 Introduction to Quivers and Properties I

Fundamental concepts of quivers and their representations will be introduced, along with a discussion of their basic properties. Additionally, we will present the associated path algebra. To help build familiarity with these ideas, illustrative examples such as L_r,K_r, and S_r will be included. Gabriel's Theorem will also be mentioned. The references for this material are [Bri12, Sections 1.1 - 1.2] and [Mar, Sections 1.1 - 1.3].

3.2 Introduction to Quivers and Properties II

The focus will be on the algebraic properties of quiver representations. We will examine indecomposable, simple, injective, and projective objects within the category of quiver representations. Additionally, we recall Jordan–Hölder filtrations, Schur's Lemma [Soi21, Section 8.1.], and discuss standard resolutions. The primary reference for this material is [Mar, Section 2], while [Bri12, Sections 1.3 - 1.4] should also be consulted for a more algebraic perspective.

4.1 Affine Moduli Spaces of Quiver Representations

The Theorem of Le Bruyn and Procesi will be presented and explained. We will define the affine GIT quotient and provide an example. Our goal is to understand the points of the affine GIT quotient that correspond to closed orbits, characterizing the associated representations through cocharacters. Special attention will be given to [Soi21, Theorem 8.2.12, Corollary 8.2.13].

4.2 Moduli Spaces of Quiver Representations

In this talk, we will construct moduli spaces of quiver representations using projective GIT, exploring how the concepts of stability and semistability apply to these representations. The Hilbert-Mumford criterion will be explained, followed by the introduction of θ-stability and a proof of [Soi21, Theorem 8.3.3]. We will also discuss the equivalent concept of slope-stability. The primary reference for this material is [Soi21, Section 8.3], while further details on the Hilbert-Mumford criterion can be found in [Hos15, Section 6.3].

5.1 Algebraic Aspects of Stability

In this talk, we explore the fundamental properties of (semi)stable representations, as discussed in [Rei08, Section 4]. We also introduce the Harder-Narasimhan filtration associated to a given representation, along with its functorial behavior, as described in [Rei08, Lemma 4.8]. Additionally, we examine how to obtain a torsion pair from this filtration, referring to [Rei08, Lemma 4.10]. These concepts are closely related to stability conditions on abelian categories, as outlined in [Rud97] and [Bay, Section 2].

5.2 Further Geometric Properties of Quiver Moduli

We now turn to the geometry of moduli spaces of quivers, where we address the existence of (semi)stable representations, the universal bundle, and coordinates. Our discussion will follow [Rei08, Section 5], and we will also compare these results with [Soi21, Thm 8.3.10] and the preceding discussion.

6.1 Framed Representations

As shown in [Soi21, Proposition 9.1.2], the moduli space of θ-stable representations is either empty or finite for many choices of the dimension vector. To address this, we introduce framed representations and associate to them a moduli space, the properties of which are described in [Soi21, Theorem 9.1.5]. We discuss the proof of [Soi21, Theorem 9.1.5] and recall key results used in the proof, such as [Soi21, Theorem 7.2.7] and [Soi21, Theorem 8.3.3]. Additionally, we will examine the examples provided in [Soi21, Section 9.2].

6.2 Symplectic Geometry and Double Quivers

We begin by introducing symplectic manifolds and Lie groups, following [Soi21, Section 10.1]. We then discuss the action of connected reductive groups on smooth affine varieties, with a particular focus on Lie algebra and Poisson varieties. Our goal is to apply these results to the study of Nakajima quiver varieties. Additionally, we introduce the symplectic structure defined by the trace pairing, as outlined in [Soi21, Section 10.3]. This material is covered in [Soi21, Section 10].

7.1 Nakajima Quiver Varieties

We will introduce Nakajima quiver varieties and provide an explanation of [Soi21, Theorem 11.1.5], followed by a discussion of an example.

7.2 Reincarnations of Quiver Moduli

Our aim is to provide a survey of the literature on quiver moduli. We will explore quiver realizations of moduli spaces of sheaves, connections, and instantons, as discussed in [Soi21, Section 12]. Time permitting, we will also review results on the geometry of quiver moduli spaces, as presented in [Rei08, Sections 6-8].

References

[Bay] Arend Bayer, *A tour to stability conditions on derived categories*, https://www.maths.ed.ac.uk/ abayer/dc-lecture-notes.pdf.

[Bri12] Michel Brion, *Representations of quivers*, Geometric methods in representation theory. I, Sémin. Congr., vol. 24-I, Soc. Math. France, Paris, 2012, pp. 103–144.

[Hos15] Victoria Hoskins, *Moduli problems and geometric invariant theory*, 2015.

[Mar] Ray Maresca, *An introduction to quiver representations.*

[Rei08] Markus Reineke, *Moduli of representations of quivers*, Trends in representation theory of algebras and related topics, EMS Ser. Congr. Rep., Eur. Math. Soc., Zürich, 2008, pp. 589–637.

[Rud97] Alexei Rudakov, *Stability for an abelian category*, J. Algebra **197** (1997), no. 1, 231–245.

[Sch14] Ralf Schiffler, *Quiver representations*, CMS Books in Mathematics/Ouvrages de Mathématiques de la SMC, Springer, Cham, 2014.

[Soi21] Alexander Soibelman, *Lecture notes on quiver representations and moduli problems in algebraic geometry*, 2021.

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