Vector bundles on curves

Program by Timo Richarz and Torsten Wedhorn

Winter term 2024/25

The goal of the seminar is to study vector bundles on algebraic curves and the geometry of their moduli stacks. The main source of the seminar are Michael Rapoport's lecture notes [Rap19; Rap20] from the courses in fall 2019 and winter 2019/20 at Maryland. The seminar is directed towards Master and PhD students with prior knowledge in Algebraic Geometry as covered by a two semester course in scheme theory.

Comments

As a general guideline, each talk should explain at least one "typical" argument in detail. The organizers leave the choice of argument to each speaker. Also, each speaker should take the freedom to refer to the notes for details depending on the available time. In addition, the organizers request that each speaker meets with one of the organizers twice before the talk: one meeting three weeks before the talk in which the speaker explain what material will be covered in the talk, and one meeting one week prior to the talk in which the speaker presents the notes. Ideally, the notes should exactly be what the speaker intends to write at the blackboard. Of course, during the actual talk the notes might differ from what is written at the blackboard but it is important to make yourself clear before your talk why, how and what you write at the blackboard.

In case you are interested in participating but not sure what talk to give, please feel always free to contact the organizers.

Time and place

The seminar takes place in a hybrid format.

- Tuesdays, 14:00 – 15:30 during the winter term 2024/25.
- Start date: October 15, end date: February 11
- There will be no talk on October 22, December 17 and the winter break from December 23 until January 10. On December 17 we meet as usual and everyone is invited to bring some cookies, hot or cold beverage of your choice and some math questions.
- The seminar will take place in Darmstadt, Room S215 401.
- Zoom meeting ID: 612 2072 7363, Password: Largest six digit prime number.

Talks

Introduction and distribution of talks (Oct 15)

This is an overview talk given by one of the organizers explaining the interrelation of talks and the structure of the seminar. In the end, we distribute the talks.

Talk 1 - Vector bundles (Oct 29)

The main source for this talk is [GW10, Sections (11.1)–(11.7)], compare with [Rap19, Section 1].

There are three different points of view on vector bundles on a scheme X:

1. as geometric vector bundles on X,

2. as locally free $\mathcal{O}_X$-modules of finite rank, and

3. as torsors under the general linear group.

More precisely, define the category of (geometric) vector bundles $\mathrm{Vec}_X^{(\mathrm{geo})}$ and show $\mathbb{V}\colon \mathrm{Vec}_X \cong \mathrm{Vec}_X^{\mathrm{geo,op}}$ with quasi-inverse the local sections functor [GW10, Proposition 11.6]. Explain that $\mathrm{Vec}(X)$ viewed as a full subcategory of all $\mathcal{O}_X$-modules $\mathrm{Mod}_{\mathcal{O}_X}$ is closed under direct sums, inner homomorphisms and tensor products. Introduce the category Bun_G of G-torsors on X [GW10, Section (11.5)]. If $G = \mathrm{GL}_{n,X}$, show that $\mathrm{Vec}_{X,n}^{\cong} \cong \mathrm{Bun}_{\mathrm{GL}_{n,X}}, \mathcal{E} \mapsto \underline{\mathrm{Isom}}(\mathcal{O}_X^n, \mathcal{E})$ [GW10, Proof of Proposition 11.15] where $\mathrm{Vec}_{X,n}$ is the category vector bundles of rank n and $\mathrm{Vec}_{X,n}^{\cong}$ its (non-full) subcategory with the same objects and isomorphisms as morphisms. Discuss the Picard group [GW10, Section (11.7)] and give examples as time permits.

Talk 2 - Algebraic curves (Nov 05)

This is an overview talk over the theory of algebraic curves. The aim is to elaborate on the material in [Rap19, Sections 1.2 and 1.4] where we refer to [GW23, Section 26] for details.

More precisely, as in [Rap19, Definition 1.2.1] we adopt the following convention for the rest of the seminar: an (algebraic) curve[1] X over an algebraically closed field k is a connected, smooth, projective k-scheme of Krull dimension 1. Note that this implies that C is a 1-dimensional regular integral scheme.

If $k = \mathbb{C}$, then $X(\mathbb{C})$ is a compact Riemann surface and every such surface arises as $X(\mathbb{C})$ for some curve X [GW23, Section (26.7)].

Introduce the dualizing (or canonical) sheaf $\omega_X = \Omega^1_{X/k}$ and the genus $g(X) := \dim_k \mathrm{H}^0(X, \omega_X) \in \mathbb{Z}_{\geq 0}$. Recall the vanishing of quasi-coherent cohomology in degrees $> \dim(X) = 1$, Serre duality and the Riemann–Roch formula.

Recall the notion of Weil divisors [GW10, Section (11.13)] and Cartier divisors [GW10, Section (11.9)] in the case of curves (see also [GW10, Sections (15.8), (15.9)], note that many definitions simplify by our assumptions on curves). Recall that Weil divisors agree with Cartier divisors [GW10, Section (11.13)] since our curves are regular and recall the exact sequences

$$0 \to k^\times \to k(X)^\times \overset{\mathrm{div}}{\to} \mathrm{Div}(X) \overset{D \mapsto \mathcal{O}_X(D)}{\to} \mathrm{Pic}(X) \to 0,$$
$$0 \to \mathrm{Pic}^0(X) \to \mathrm{Pic}(X) \overset{\deg}{\to} \mathbb{Z} \to 0.$$

Discuss curves of low genus $g = 0, 1$ as in [Rap19, Section 1.2] and give some examples and proofs [GW23, Sections (26.16) and (26.17)] as time permits.

Talk 3 - Construction of vector bundles (Nov 12)

Follow [Rap19, Section 1.4] and discuss general properties of coherent sheaves on curves, in particular the criterion in Lemma 1.4.1 for being a vector bundle and the generalization of the Riemann Roch formula from line bundles to coherent sheaves in Theorem 1.4.3.

Then follow [Rap19, Chapter 2] and explain how to construct new vector bundles from given ones via constructions from linear algebra, lattice constructions and by modifications. In particular, explain Lemma 2.2.4, Corollary 2.2.7, Proposition 2.3.1 and Corollary 2.4.3 as time permits.

[1] The notion of curve in [GW23, Section 26] is much more general and some statements there simplify for our more restrictive definition of curves.

Talk 4 - Vector bundles on curves of low genus (Nov 19)

Explain the classification of vector bundles on curves of genus 0, and of genus 1 as time permits following [Rap19, Chapter 3], see also [GW23, Sections (26.22) and (27.50)] for another approach.

Talk 5 - Harder–Narasimhan filtration (Nov 26)

The aim of this talk is to explain the construction of the Harder–Narasimhan filtration on vector bundles on a curve [GW23, Theorem 26.156] (see also [Rap19, Theorem 4.1.16]). For this follow [GW23, Sections (26.24) and (26.25)] (see also [Rap19, Section 4.1]), introduce the slope, (semi-)stable vector bundles and proof some basic facts.

Talk 6 - Existence of semi-stable vector bundles and examples (Dec 03)

State [Rap19, Theorem 4.2.6] and sketch its proof. Mention the result for stable vector bundles. Discuss curves of genus 0 and 1: the HN filtration splits [GW23, Proposition 26.161]; classify (semi-)stable vector bundles [Rap19, beginning of Section 4.2].

Talk 7 - Vector bundles in families (Dec 10)

This follows [Rap19, Sections 6.1 and 6.2]. The aim of the talk is to sketch the proof that the HN polygon goes up under specialization [Rap19, Theorem 6.2.1]. In particular, introduce the S-families of vector bundles, show that the HN filtration is compatible with change of the base field (see also [GW23, Theorem 26.159]), skip the discussion on page 52, introduce the specialization relation and the partial order on HN vectors. A sketch of [Rap19, Proposition 6.1.14] would be nice as well.

Talk 8 - Stacks and examples (Jan 14)

For this talk some prior exposure to the theory of stacks is helpful. Also, we recommend [Hei10] for a first reading. Define the notion of a stack on sites following [Alp14, Section 2.5]. Explain how to compute fiber products. Later we will focus on the étale site on the category of schemes. Give (quasi-)coherent sheaves and vector bundles as examples, see Example 2.5.9.

Talk 9 - Algebraicity of $\mathrm{Bun}_{X,n}$ (Jan 21)

We recommend [Hei10] for a first reading. Define the notion of an algebraic stack following [Alp14, Definition 3.1.6], mention quotient stacks in Theorem 3.1.10 and prove Theorem 3.1.21. The existence of Quot schemes can be used as a blackbox. Please feel free to extend the results to stacks of G-bundles for smooth affine k-group schemes G, see [Rap20, Section 2.2].

Talk 10 - Geometry of $\mathrm{Bun}_{X,n}$ (Jan 28)

The main references are [Hei98, Section 2.1] and private communication with the organizers. Show that $\mathrm{Bun}_{X,n}$ is smooth. Discuss the connected components of $\mathrm{Bun}_{X,n}$, indexed by the degree of a vector bundle. Define the notion of a stratification on an algebraic stack. Show that the HN-filtration defines a decomposition of $\mathrm{Bun}_{X,n}$ indexed by the partially ordered set of HN-polygons. Show that the stable and the semistable locus is open in $\mathrm{Bun}_{X,n}$. If time remains, discuss the geometry of $\mathrm{Bun}_{\mathbb{P}^1_k,n}$.

Talks 11 & 12 (Feb 4 & 11)

The precise material depends on the interest of the speakers. Possible topics include adequate moduli spaces a geometric invariant theory, uniformization or its adelic version, affine Grassmannians, classification of line bundles, Betti cohomology of $\mathrm{Bun}_{X,n}$ and so on. Please feel free to talk with the organizers at any point.

References

[Alp14] Jarod Alper. 'Stacks and Moduli'. working draft available online. 2014.

[GW10] Ulrich Görtz and Torsten Wedhorn. *Algebraic geometry I*. Advanced Lectures in Mathematics. Schemes with examples and exercises. Vieweg + Teubner, Wiesbaden, 2010, pp. viii+615. ISBN: 978-3-8348-0676-5. DOI: 10.1007/978-3-8348-9722-0. URL: https://doi.org/10.1007/978-3-8348-9722-0.

[GW23] Ulrich Görtz and Torsten Wedhorn. *Algebraic geometry II: Cohomology of schemes—with examples and exercises*. Springer Studium Mathematik—Master. Springer Spektrum, Wiesbaden, [2023] ©2023, pp. vii+869. ISBN: 978-3-65843-030-6; 978-3-65843-031-3. DOI: 10.1007/978-3-658-43031-3. URL: https://doi.org/10.1007/978-3-658-43031-3.

[Hei10] Jochen Heinloth. 'Lectures on the moduli stack of vector bundles on a curve'. In: *Affine flag manifolds and principal bundles*. Trends Math. Birkhäuser/Springer Basel AG, Basel, 2010, pp. 123–153. ISBN: 978-3-0346-0287-7. DOI: 10.1007/978-3-0346-0288-4_4. URL: https://doi.org/10.1007/978-3-0346-0288-4_4.

[Hei98] Jochen Heinloth. 'Über den Modulstack der Vektorbündel auf Kurven'. working draft available online. 1998.

[Rap20] Michael Rapoport. 'Moduli spaces of vector bundles'. Lecture notes available upon request. 2019/20.

[Rap19] Michael Rapoport. 'Vector bundles on algebraic curves'. Lecture notes available upon request. 2019.