Congruence modules and the Wiles–Lenstra–Diamond numerical criterion in higher codimension, after Iyengar–Khare–Manning

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The seminar will take place on Fridays, 9-11, in room SR8.

We will study the paper [IKM22]. In the following, all references are to this paper, unless otherwise stated.
In the paper, the authors develop some commutative algebra tools that allow them to generalize the "numerical criterion" of Wiles–Lenstra–Diamond, in order to prove *integral* $R = \mathbb{T}$ theorems in the case when a suitably defined *Wiles defect* is positive. This generalization parallels the already known extension of the patching method to cases of positive defect by Calegari–Geraghty. The authors give two examples of arithmetic applications of their result, proving integral $R = \mathbb{T}$ results for Hecke algebras over PGL_2 of a number field (conditionally on Langlands-type conjectures), and for weight 1 Hecke eigensystems coming from the cohomology of Shimura curves (unconditionally).
Below is a tentative program. The first 5 talks consist purely of commutative algebra, and require no arithmetic input. Talks 6 and 7 provide the necessary input from the patching method and the Galois deformation theory. Talks 8 and 9 give the setting for the two arithmetic applications, and talk 10 deals with the proof of the main modularity result.

0. (15/11) Overview

1. (22/11) Congruence modules and Wiles defect [Sections 2,3]

Explain how to attach a *congruence module* $\Psi_A(M)$ to a finitely generated module M over a Noetherian local ring equipped with a so-called *augmentation*, i.e. a surjective local homomorphism $\lambda_A\colon A \to \mathcal{O}$ to a discrete valuation ring $\mathcal{O}$. Prove Theorem 2.5. Show that this notion of congruence module recovers the classical one in the case of depth 0 (Proposition 2.10).
Define the *Wiles defect* $\delta_A(M)$ of M. Show the implications in the diagram of Section 3.3. Prove the defect formula of Lemma 3.7.
One can consult [BH98] for many of the commutative algebra inputs.

2. (29/11) Cohen–Macaulay modules and complete intersection rings [Sections 4,5]

Make reminders about dualizing complexes and Cohen–Macaulay modules, and prove Proposition 4.4. Show how to describe the congruence module of a Cohen–Macaulay module as the cokernel of the adjoint of a map of Ext groups (Proposition 4.7). Prove that the cokernel appearing in the defect formula from the previous talk is an Ext group.
Assuming that A is complete intersection, one can find another complete intersection C with a particular shape, with a map $C \to A$ that identifies cotangent modules and, in particular, their torsion parts Φ_C and Φ_A. Give the criterion for detecting when a map of complete intersections is an isomorphism from the torsion in their cotangent modules (Lemma 5.10).
As for the previous section, [BH98] is a useful complementary reference.

3. (6/12) Tate resolutions [Section 6]

The material of this section is essentially extracted from Avramov's book [Avr10, Section 6]. Explain the Tate construction of a resolution of a complete noetherian local ring A, and use it to describe the graded $\mathcal{O}$-algebra $F_A^*(\mathcal{O})$ (i.e the torsion part of the graded $\mathcal{O}$-algebra $\mathrm{Ext}_A^*(\mathcal{O}, \mathcal{O})$) for $A \in \mathcal{C}_O(c)$ (Theorem 6.8). Give the criterion for detecting when $A \to B$ induces an isomorphism $F_B^*(\mathcal{O}) \cong F_A^*(\mathcal{O})$ (Proposition 6.10), and the criterion for detecting complete intersections from the Tate resolution (Proposition 6.11). Section 6.12 can be skipped, as it reappears later with more details.

4. (13/12) Congruence modules and Wiles defect under surjections [Sections 7,8]

Prove Theorem 7.4 about the invariance of the length of congruence modules along a surjection $A \to B$ in $\mathcal{O}_{\mathcal{O}}(c)$, and deduce Corollary 7.5. Prove the invariance of the defect δ under quotienting by a non-zero divisor (Theorem 8.2), via the relation between the congruence modules in the same setting (Lemma 8.7).

5. (20/12) Wiles defect and free direct summands [Section 9]

Prove the criterion for detecting from the Wiles defect when a Cohen–Macaulay module over a Gorenstein ring has a free direct summand (Theorem 9.2). Prove that a ring in $\mathcal{O}(c)$ is complete intersection if and only if its Wiles defect vanishes (Theorem 9.5), and the characterization of modules of Wiles defect 0.

6. (10/1) Patching [Sections 10,11]

Recall the patching constructions, following Section 10 and if necessary the references therein. In particular, state and prove Theorem 10.6 on the construction of the patching functor from the category of "patching systems". Unless time allows, you can state without proof Theorem 11.3 on the compatibility of the patching construction with duals (Theorem 11.3). One can consult [CG18] for more background on the patching method.

7. (17/1) Galois deformation conditions [Section 12]

(We might organize a preliminary talk on Galois deformations on the Wednesday before the seminar.) Define minimally ramified local deformation rings, and recall the structure result Proposition 12.1. Use the local conditions to define the relevant global deformation rings as in Section 12.2, and give Proposition 12.3. Various definitions and statements are taken from [CHT08].

8. (24/1) Hecke eigensystems and Galois representations for PGL_2 [Section 13]

Define the orbifold Y_K attached to the algebraic group PGL_2 over a number field F and to a compact open subgroup $K \subset \mathrm{PGL}_2(\mathbb{A}_F^\infty)$ (this corresponds to case (PGL2) from the beginning of Part 3 of the paper). Define the Hecke algebra of this level, and state Conjecture A attaching Galois representations to Hecke eigensystems. Rephrase it as an isomorphism in a suitable derived category as in Proposition 13.6. For various statements concerning this setting, the speaker can consult [NT16].

State Conjectures B (on the properties of the representations produced from Conjecture A), C (on the vanishing of homology groups in certain degrees), and D (on the surjectivity of the level-lowering map on homology groups).

9. (31/1) Weight 1 Hecke eigensystems and Galois representations for quaternion algebras [Section 14]

This is the analogue of Section 13 in the case of weight 1 Hecke eigensystems for a ramified quaternion algebra $\mathfrak{D}$ (case (Wt1) from the beginning of Part 3 of the paper). Here Y_K is replaced by the Shimura curve $X_K^{\mathfrak{D}}$. Explain the setting and prove Theorem 14.5 (the analogue of Conjecture A), Theorem 14.6 (the analogue of Conjecture B; only sketch the proof), and the analogues of Conjectures C and D (see Section 14.3). Announce Theorem 14.9 and prove its corollary, the mod ℓ modularity Theorem 14.10. A complementary reference in this setting is [Box+21].

10. (7/2) Modularity [Section 15]

Prove the main modularity result, Theorem 15.1.

References

[Avr10] Luchezar L. Avramov. "Infinite free resolutions". In: *Six lectures on commutative algebra*. Mod. Birkhäuser Class. Birkhäuser Verlag, Basel, 2010, pp. 1–118.

[BH98] Winfried Bruns and H. Jürgen Herzog. *Cohen-Macaulay rings*. Vol. 39. Cambridge Studies in Advanced Mathematics. Cambridge University Press, 1998.

[Box+21] George Boxer et al. "Abelian surfaces over totally real fields are potentially modular". In: *Publ. Math. Inst. Hautes Études Sci.* 134 (2021), pp. 153–501.

[CG18] Frank Calegari and David Geraghty. "Modularity lifting beyond the Taylor-Wiles method". In: *Invent. Math.* 211.1 (2018), pp. 297–433.

[CHT08] Laurent Clozel, Michael Harris, and Richard Taylor. "Automorphy for some ℓ-adic lifts of automorphic mod ℓ Galois representations". In: *Publ. Math. Inst. Hautes Études Sci.* 108 (2008), pp. 1–181.

[IKM22] Srikanth B. Iyengar, Chandrashekhar B. Khare, and Jeffrey Manning. *Congruence modules and the Wiles–Lenstra–Diamond numerical criterion in higher codimension.* 2022. URL: `https://arxiv.org/abs/2206.08212`.

[NT16] James Newton and Jack A. Thorne. "Torsion Galois representations over CM fields and Hecke algebras in the derived category". In: *Forum Math. Sigma* 4 (2016), Paper No. e21, 88.