24.10.2024

Seminar: The direct summand theorem

AG Venjakob

Winter semester 24/25

Organisation:	We meet every Thursday at 11:15 in SR 8 (Mathematikon). Online participation is possible - please contact us for the Zoom data.
Contact:	Marlon Kocher, mkocher@mathi.uni-heidelberg.de, Marvin Schneider, mschneider@mathi.uni-heidelberg.de

Overview

In this seminar, we follow the lecture notes of Bhatt [Bh17] to learn about almost mathematics, perfectoid spaces and their application in the direct summand theorem (DST):

Theorem (André). *Let $i : A \hookrightarrow B$ be a finite extension of noetherian rings. Assume that A is regular. Then the inclusion i splits as a morphism of A-modules.*

In 2016 André published a preprint proving the DST (back then a conjecture) using a perfectoid Abhyankar lemma (cf. [And18a]; [And18b]). Shortly afterwards, Bhatt significantly simplified the proof (cf. [Bh18]), using only a key construction of André's, but circumventing the full perfectoid Abhyankar lemma. He also proved a derived variant:

Theorem (Bhatt). *Let A be a regular noetherian ring and $X \to \operatorname{Spec} A$ a proper surjective map of schemes. Then $A \to \mathbf{R}\Gamma(X, \mathcal{O}_X)$ splits in the derived category of A-modules.*

Both proofs heavily rely on the application of perfectoid spaces and the strategy is often summarized as follows: Using perfectoid geometry, construct a (very large) ring extension $R|A$, which is almost faithfully flat and where R is perfectoid. Afterwards, show that the splitting problem has an almost solution after base change to R and finally descend the almost solution to an actual solution. In this seminar, we develop all the necessary theory about almost ring theory, Huber's adic and Scholze's perfectoid spaces from scratch. We will prove a few of the foundational results of perfectoid spaces of [Sch12] before finally giving Bhatt's proof of the direct summand theorem.

The Talks

Our main source will be the lecture notes of Bhatt [Bh17]. We want to point out, however, that the original articles are also very accessible and that a *Hot Topics* workshop from 2018 on the very same subject produced excellent notes and interesting videos (cf. [BINS]).

The aim is to present the topics in as much detail as possible and not just provide a rough overview. If necessary, 1.5-2 sessions are available for each of the talks.

Previous knowledge about adic and perfectoid spaces is not required for this seminar; we will cover these topics in detail. Feel free to join the seminar if you want to attend an introduction to these techniques.

Talk 1: Almost modules and almost commutative algebra – Immanuel Klevesath (17.10.)

Go through [Bh17, §4.1 - 4.2]: Introduce the categories of almost R-modules [Bh17, Proposition 4.1.7]. Show, that the localization functor $\mathrm{Mod}_R \to \mathrm{Mod}_R^a$, $M \mapsto M^a$ admits a right adjoint, namely $M \mapsto M_* := \mathrm{Hom}_R(I, M)$ and a left adjoint given by $M \mapsto M_! := I \otimes_R M_*$. It can be helpful to also take a look at [GR, §2.2].

Talk 2: Almost étale extensions and the almost purity theorem in characteristic p – Nils Tietke (24.10.)

Go through [Bh17, §4.3] and introduce *almost finite étale maps* [Bh17, Definition 4.3.1]. Show the almost purity theorem in characteristic p [Bh17, Theorem 4.3.6].

Talk 3: Non-Archimedean Banach Algebras – Nils Witt (31.10.)

There are multiple equivalent definitions of perfectoid algebras, one of which uses ultrametric Banach algebras. This talk, covering §5 in [Bh17], is supposed to give an overview of the subject and explain how the category of uniform Banach algebras has a more algebraic description.

Talk 4: Perfectoid algebras – Anna Blanco Cabanillas (7.11.)

This talk serves as an overview about perfectoid algebras. Explain [Bh17, §6] in more detail, in particular the tilting correspondence ([Bh17, Corollary 6.2.6]) and the almost purity theorem ([Bh17, Theorem 6.2.10]), which we are going to prove in talk 9.

Talk 5: Adic spaces I – Lars Wüste-Schmülling (14.11.)

The "correct" topological framework for perfectoid spaces are Huber's adic spaces. Develop the necessary background in order to introduce adic spaces in the subsequent talk ([We, §1] or [Bh17, §7] are good references): Define valuation rings and their spectra. Introduce Huber rings and explain their topology. Define Huber pairs and their associated affinoid adic spaces.

Talk 6: Adic spaces II – Yanik Kleibrink (21.11.)

Explain, without giving a detailed proof, the main result of [Bh17, §7.4], i.e. a basis of quasi-compact opens of an affinoid adic space is given by rational subsets. Go on and introduce the structure presheaf associated to an affinoid adic space and discuss it in more detail ([Bh17, §7.5]). Finally introduce the notion of a general adic space.

Talk 7: Perfectoid spaces I: Tilting of Rational Subsets – Marlon Kocher (28.11.)

Perfectoid spaces are adic spaces that are locally affinoid adic spaces of perfectoid rings. Go through [Bh17, §9.1 – 9.2], introduce them and show their basic properties. The goal of this talk should be to prove [Bh17, Theorem 9.2.2].

Talk 8: Perfectoid spaces II: Tate acyclicity – Marvin Schneider (5.12.)

While for a ring R the natural presheaf of its spectrum $\operatorname{Spec} R$ is always a sheaf, this fails in general for the adic spectrum associated to a Huber pair (R, R^+). However, perfectoid K-algebras (R, R^+) over a perfectoid field K are sheafy ("Tate acyclicity for perfectoids", [Bh17, Theorem 9.3.1]). Goal of this talk is to sketch a proof of this theorem.

Talk 9: The almost purity theorem – Christian Dahlhausen (12.12.)

With the tools developed so far, we can show one of the key ingredients to Bhatt's proof of the DST, namely a perfectoid version of Faltings's almost purity theorem. This is done in [Bh17, §10].

Talk 10: The direct summand theorem – Max Witzelsperger (19.12.)

Finally we can prove the direct summand theorem. Following André, one constructs a suitable ring extension and shows the DST after base change to this ring with a "Riemann Hebbarkeitssatz," which one then descends afterwards. In [Bh17, §11], only the non-derived variant is proved there, so consult [Bh18] as well, if time permits.

References

[And18a] André, Y., *"La conjecture du facteur direct."*, *Publications Mathématiques De L'IHÉS.* **127**, 71-93, `https://doi.org/10.1007/s10240-017-0097-9`

[And18b] André, Y., *"Le lemme d'Abhyankar perfectoide."*, *Publications Mathématiques De L'IHÉS.* **127**, 1-70, `https://doi.org/10.1007/s10240-017-0096-x`

[Bh17] Bhatt, Bhargav (2017), *"Lecture notes for a class on perfectoid spaces"*, `https://websites.umich.edu/~bhattb/teaching/mat679w17/lectures.pdf`

[Bh18] Bhatt, Bhargav (2018), *"On the direct summand conjecture and its derived variant"*, *Inventiones Mathematicae.* **212**, 297-317, `https://doi.org/10.1007/s00222-017-0768-7`

[BINS] Bhatt, Bhargav, Iyengar, Srikanth ,Nizioł, Wiesława, and Singh, Anurag ., *"Hot Topics: The Homological Conjectures"*, `https://www.msri.org/workshops/842`

[GR] Gabber, Ofer and Ramero, Lorenzo (2002), *"Almost ring theory"*, `https://arxiv.org/pdf/math/0201175`

[Sch12] Scholze, Peter, *" Perfectoid Spaces." Publications Mathématiques De L'IHÉS.* **116**, 245-313, `https://doi.org/10.1007/s10240-012-0042-x`

[We] Wedhorn, Torsten (2019), *"Adic spaces"*, `https://arxiv.org/pdf/1910.05934`