GAUS-AG: Mochizuki's proof of the Hom-conjecture

Organizers: Alexander Schmidt[*], Jakob Stix[†]
Program proposal: Magnus Carlson[‡], Ruth Wild[§]

Introduction

The goal of this AG is to understand the proof of the following fundamental result of Mochizuki in anabelian geometry, known as the *Hom-conjecture* for sub p-adic fields:

Theorem ([Moc99, Theorem A]). *Let K be a sub p-adic field, i.e., a subextension of a finitely generated field extension of $\mathbb{Q}_p$, and denote by G_K the Galois group of K. Let X, Y be smooth, geometrically irreducible projective curves over K, with Y hyperbolic. Then the natural map*

$$\mathrm{Hom}_K^{\mathrm{dom}}(X, Y) \to \mathrm{Hom}_{G_K}^{\mathrm{open}}(\pi_1(X), \pi_1(Y))$$

is bijective, where the right hand side is open homomorphisms up to conjugation by an element of $\pi_1(Y_{\overline{K}})$.

Grothendieck conjectured this result in his letter to Faltings [Gro97] in the case when K is a finitely generated extension of $\mathbb{Q}$. Recall the Tate conjecture for a field K finitely generated over $\mathbb{Q}$: given abelian varieties A and B and a prime ℓ, the natural map

$$\mathrm{Hom}(A, B) \otimes \mathbb{Z}_\ell \to \mathrm{Hom}_{G_K}(T_\ell(A), T_\ell(B))$$

is bijective, where $T_\ell(A)$ and $T_\ell(B)$ are the ℓ-adic Tate modules of A and B.

The similarity between the Tate conjecture and the Hom conjecture might lead one to assume that Theorem A is an essentially *global* result, and that a version of the Hom conjecture over p-adic fields is unlikely. However, in the words of Mochizuki [Moc99, pg. 326]: "The reason it took so long for Theorem A to be discovered was the overwhelming prejudice of most people in the field that the Grothendieck Conjecture for hyperbolic curves is an essentially global result, akin to the Tate Conjecture for abelian varieties over number fields. In fact, however, it is much more natural to regard the Grothendieck Conjecture for hyperbolic curves as an essentially local, p-adic result that belongs to that branch of arithmetic geometry known as p-adic Hodge theory".

[*]schmidt@mathi.uni-heidelberg.de
[†]stix@math.uni-frankfurt.de
[‡]carlson@math.uni-frankfurt.de
[§]wild@math.uni-frankfurt.de

The basic strategy of Mochizuki's proof is easy to explain. Suppose that X and Y are hyperbolic and not hyperelliptic. Given an open map $f: \pi_1(X) \to \pi_1(Y)$ over G_K, we can, for the purpose of proving Theorem A, assume that f is surjective. Since X and Y are étale $K(\pi, 1)$-spaces, we have a natural injective map $f^*: H^1(Y, \mathbb{Q}_p) \to H^1(X, \mathbb{Q}_p)$ between étale cohomology groups. After tensoring with $\mathbb{C}_p$, the completion of an algebraic closure of $\mathbb{Q}_p$, p-adic Hodge theory provides a G_K-equivariant map

$$H^1(Y, \mathcal{O}_Y) \otimes \mathbb{C}_p \oplus H^0(Y, \Omega_Y) \otimes \mathbb{C}_p(-1) \to H^1(X, \mathcal{O}_X) \otimes \mathbb{C}_p \oplus H^0(X, \Omega_X) \otimes \mathbb{C}_p(-1).$$

Twisting by $\mathbb{C}_p(1)$ and taking Galois invariants then gives a map $H^0(Y, \Omega_Y) \to H^0(X, \Omega_X)$. Thus, $f: \pi_1(X) \to \pi_1(Y)$ induces naturally a rational map $f^*: \mathbb{P}(\Omega_X) \to \mathbb{P}(\Omega_Y)$ between the associated projective spaces. Since X and Y are not hyperelliptic, they embed canonically into $\mathbb{P}(\Omega_X)$ and $\mathbb{P}(\Omega_Y)$ respectively. Thus, to construct from f a map of schemes, it suffices to show that $f^*: \mathbb{P}(\Omega_X) \to \mathbb{P}(\Omega_Y)$ "preserves relations"', i.e. if $a \in \oplus_{n\geq 0} H^0(Y, \Omega_Y)^{\otimes n}$ is an element which vanishes when "restricted to Y", the pulled back element $f^*(a)$ vanishes when restricted to X. Mochizuki's proof of the preservation of relations is highly innovative and technical, and the reader interested in the details should attend the AG.

In the proof of Mochizuki's result, we will make ourselves acquainted with the following notions, all interesting in their own right :

- p-adic Hodge theory
- the very basics of Faltings' "almost mathematics"
- unipotent fundamental groups
- Bloch–Kato Selmer groups
- p-divisible groups
- the Tate conjecture
- ⋯

Description of the talks

Each talk should last 90 minutes. In this seminar, we following Faltings' exposition [Fal98], and all unspecified references are to it. Throughout, K will usually be a finite extension of $\mathbb{Q}_p$ with absolute Galois group G_K, and X usually a smooth projective curve over K.

Day I: Introduction

Talk 1: Introduction *(Jakob)*.
The goal of this talk is to give an overview of the strategy of Mochizuki's proof, as well as to give a broad introduction to the fundamental conjectures in anabelian geometry. Following, for example, [NTM01], define the *Hom*, *Isom* and the *section* conjecture. Mention how Grothendieck only conjectured it for finitely generated extensions of $\mathbb{Q}$. State then the Hom-conjecture for sub p-adic fields, and remark how Mochizuki expanded the fields over which "anabelian phenomena" could be expected to hold. Then sketch the general strategy

of Mochizuki's proof, some possible references are the introduction to [Moc99], and [NTM01, §5.2]. Make sure to emphasize the crucial inputs needed from p-adic Hodge theory.

Talk 2: Galois cohomology and Hodge-Tate decomposition *(Jonathan)*.
In this talk, we will cover pg. 136-137 of [Fal98], to the end of Theorem 4. The first goal is to calculate the Galois cohomology groups on pg. 136. Start by introducing basic notions from almost mathematics, and then proceed by stating the almost purity theorem. As a corollary, one finds that "finite étale covers give almost split almost finite étale covers" ([Bha17, Prop.10.0.9]). If time permits, state how to deduce this proposition from Theorem 10.0.1 of *ibid*. Explain how this can be used to get the almost vanishing of the higher Galois cohomology of $\mathcal{O}_{\mathbb{C}_K}$, and how this recover Tate's calculation of the Galois cohomology of $\mathbb{C}_K$. Then calculate the Galois cohomologies Faltings gives at the bottom of pg. 136.

Finally, state the general version of the Hodge-Tate and de Rham decomposition, possibly also for local systems (c.f. [Sch13, Thm.8.4]), and define the notions of Hodge-Tate and de Rham representations of G_K (see [BC09] for more background on period rings and Fontaine's formalism).

Additional quick summaries that work towards almost purity include [Mat18] and [Bha14]. Probably [Ols09, §3] can also be helpful.

Day II: Unipotent fundamental groups etc

Talk 3: Unipotent Tannakian categories *(Leonie)*.
This part is a bit of a digression, since none of the results are explicitly mentioned in [Fal98]. For a Tannakian category $\mathcal{T}$, define when $\mathcal{T}$ is *unipotent* (c.f. [Bet23b, Def.3.2]). For ω a fiber functor on $\mathcal{T}$, give the explicit construction of its pro-representing object $(E_{\mathcal{T},\omega}, e_{\mathcal{T},\omega})$ ([Bet23b, Prop.3.6], following [Had11, Thm.2.1]). Then briefly mention the criterion for isomorphy from [Bet23b, Prop.3.9].

Next, define the category $\mathcal{C}_{\mathrm{uni}}$ of unipotent vector bundles on X ([Had11, §2]). For $x \in X(R)$, let ω_x be the fiber functor on $\mathcal{C}_{\mathrm{uni}}$ that maps a unipotent vector bundle $\mathcal{E}$ to its stalk $\mathcal{E}_x$ at x. Give the explicit description of $(E_{\mathcal{C}_{\mathrm{uni}},\omega_x}, e_{\mathcal{C}_{\mathrm{uni}},\omega_x})$ as in [Had11, Prop.2.6].

We also need the $\mathbb{Q}_p$*-pro-unipotent étale fundamental group* $\pi_1^{\mathbb{Q}_p}(X,x)$ of X. This is constructed in a similar way, but now for the category $\mathrm{Loc}^{\mathrm{uni}}_{\mathbb{Q}_p}(X_{\text{ét}})$ of unipotent $\mathbb{Q}_p$-local systems on $X_{\text{ét}}$. Briefly explain this construction, and that it recovers the construction of $\pi_1^{\mathbb{Q}_p}(X,x)$ via Malcev completion, following [Bet23a, Lec.6].

Talk 4: Construction of $\mathfrak{h}$ and Hodge-Tateness of rational sections.
The goal of this talk is to prove that sections corresponding to rational points are *Hodge–Tate*, a notion we define in this talk . Let $G = \pi_1^{\mathbb{Q}_p}(X_{\bar{K}}, \bar{x})$, for a basepoint $\bar{x} \in X(\bar{K})$. Briefly explain in what sense G is determined by its Lie algebra $\mathfrak{g}$. Now any section $s:\ \mathrm{Gal}(\bar{K}/K) \to \pi_1(X)$ induces a $\mathrm{Gal}(\bar{K}/K)$-action on G and $\mathfrak{g}$. Explain how to get the $\mathrm{Gal}(\bar{K}/K)$-invariant filtration $E^m(\mathfrak{g}_{\mathbb{C}_K}/Z^n(\mathfrak{g}_{\mathbb{C}_K}))$ by Lie ideals, and why it is independent of the section s. At this point, we diverge from the proof of Faltings, and instead use more modern machinery. Proceed as follows: assume that the $\mathrm{Gal}(\bar{K}/K)$-action on $\mathfrak{g}_{\mathbb{C}_K}$ induced by a section turns $\mathfrak{g}_{\mathbb{C}_K}$ into a "pro-Hodge-Tate" representation. In this case, one can construct a "filtration by weights" on

$\mathfrak{g}_{\mathbb{C}_K}$. Explain how the choice of a different Galois section changes the $\mathrm{Gal}(\bar{K}/K)$-action on $\mathfrak{g}_{\mathbb{C}_K}$, and deduce that this leaves the filtration invariant. So to construct $\mathfrak{h}$, it is only necessary to find one section that turns $\mathfrak{g}_{\mathbb{C}_K}$ into a Hodge-Tate representation. The first half of the talk is discussed on [Fal98, pg.138], the second half has to be worked out.

To conclude the talk, we have to show that the action on $\mathfrak{g}_{\mathbb{C}_K}$ is indeed "pro-Hodge-Tate" when induced by a rational point. For that, give a brief overview of the main argument in [Bet23b], which actually shows that the action on $\mathfrak{g}_{\mathbb{C}_K}$ is even "pro-de-Rham". In particular, it would be nice if we could see how arguments with the pro-representing objects from Talk 3 are used, and what properties of the Riemann-Hilbert-type functor $\mathcal{RH}$ from [LZ16] are needed.

Preparing this talk requires probably some time, but Magnus and Ruth gladly share more details.

Day III: Hodge-Tate sections

Talk 5: Bloch-Kato Selmer groups.
Following [Bel09, §2.2], introduce the Bloch-Kato Selmer group $\mathrm{H}^1_f(G_K, V)$. Then prove [Bel09, Prop. 2.10]. Introduce the variants $\mathrm{H}^1_e(G_K, V)$ and $\mathrm{H}^1_g(G_K, V)$ as in [Bel09, 2.2.2]. Work out [Bel09, Exercise 2.24], assuming good reduction if necessary. Then, state [BK07, Example 3.11]. If time permits, summarize what additional work has to be done to get this result for arbitrary abelian varieties (and not just elliptic curves).

Talk 6: Hodge-Tate sections are geometric up to torsion.
Let $J^{(1)}$ be the classifying space of degree one line bundles on X. Define the notion of when a section $s_{\mathrm{ab}}: G_K \to \pi_1(J^{(1)})$ is *Hodge-Tate*, and work out the relation to the notion of Hodge-Tate sections $s: G_K \to \pi_1(X)$ from Talk 4. Show in particular that when a section s is induced by a rational point, that the section s_{ab} is Hodge-Tate. Then we prove Proposition 9: first, use [BK07, Example 3.11] to see that there is a m such that $J^{(m)}(K) \neq \varnothing$. Then explain why "lifting sections" implies "lifting points", and why this finishes the proof. Finally, give Definition 10, and summarize the paragraph right below it.

Day IV: Sections that are geometric up to torsion

Talk 7: Preparations for Proposition 11: p-divisible groups *(Benjamin)*.
In this talk, we will define p-divisible groups and prove the necessary results concerning them which are needed for Proposition 11. Our main goal is to state and prove the extension theorem [Tat67, Theorem 4]. We primarily follow the classical paper [Tat67].

Start, as in [Tat67, §2] , by defining a *p-divisible group* and homomorphisms between p-divisible groups. Proceed by giving natural examples of p-divisible groups, see [Sti09, Example 9.2] for example. Next, define as in [Tat67, §2.4] , the Galois modules $\Psi(G)$ and $T(G)$. Explain how these Galois modules relate to the generic fiber of G. Finally, prove [Tat67, Theorem 4].

Talk 8: Preparations for Proposition 11: the Tate conjecture .
Define an *semiabelian* variety A over an arbitrary base S ([FC90, Definition 2.3]). Then prove

[Fal98, Theorem 12] in as much detail as possible. The proof in Faltings is quite dense, so it is recommended to consider other sources, such as [FC90, Ch. V, Proof of Thm 4.7] or [Amb15, Chapter 1] to fill in details.

Day V: Proof of main result

Talk 9: Sections geometric up to torsion.
First, state Definition 10 of [Fal98], and then proceed by proving Proposition 11. If time permits motivate the notion of Raynaud extensions. For example, explain that while a semiabelian variety A over a base need not be an extension of an abelian scheme by a torus, the Raynaud extension G has this property. Further, G and A define the same p-adic formal scheme. In the unlikely event that there is more time, sketch the construction of Raynaud extensions, but leave out the proof of algebraization.

Explain, if needed in the Proof of Proposition 11, how the n-torsion subgroups schemes $G[n]$ and $A[n]$ relate to each other: [Gro72] is a good source for the above material on Raynaud extensions. Then show that it is enough to characterize $G_i(V^+) \otimes \mathbb{Z}_p$ inside $H^1(\mathrm{Gal}(\overline{K^+}/K^+), T_p(G))$ and give the characterization: here one can either use Ext of p–divisible groups, or fppf-cohomology. Since we have already proven the two theorems of Tate, the rest of the proof should be straightforward.

Talk 10: Proof of the Hom conjecture.
Prove Theorem 14. Start by giving the necessary background to the theorem as covered on pg. 146-147. Then show that it is enough to pass to finite extensions of K. Then explain how a non-constant K^+-point of X induces via the map of fundamental groups a non-constant K^+-point of Y: refer freely to the material in Talk 2. Then explain how this shows that the induced map $H^0(Y, \omega_Y) \to H^0(X, \omega_X)$ "preserves relations", thus inducing a map $f : X \to Y$. Conclude that by iterating this recipe on étale covers, one shows that f induces the given map on fundamental groups. Here, the discussion just before [Moc99, Theorem 14.1] might be helpful for further details on the argument.

Talk 😜. 🍜 🍸: Now let's celebrate the successful seminar!

Technical details

- We meet on five Thursdays scattered throughout the semester, from 2-6pm. The meetings alternate between Heidelberg and Frankfurt. We can also stream via Zoom.
- Each meeting consists of two 90 minute talks, with a 30 minute coffee break in between.

The precise dates and locations are as follows:

- Location Frankfurt: Room 309, Robert-Mayer Strasse 6-10
- Location Heidelberg: tba.
- Schedule:

	Day	Location
October	31	Frankfurt
November	21	Heidelberg
December	19	Frankfurt
January	16	Heidelberg
	30	Frankfurt
February	06	Heidelberg (Backup)

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