Universality of the Galois action on the fundamental group of $\mathbb{P}^1 - \{0, 1, \infty\}$

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Introduction

Belyi proved in 1979 [Bel79] the remarkable result that a connected smooth projective curve over $\mathbb{C}$ is defined over $\overline{\mathbb{Q}}$ if and only if it admits a finite map to $\mathbb{P}^1$ branched at most above $0, 1$ and ∞. This result has the implication that the natural morphism $G_{\mathbb{Q}} \to \mathrm{Out}(\pi_1(\mathbb{P}^1_{\overline{\mathbb{Q}}} \setminus \{0, 1, \infty\}))$, from the absolute Galois group of $\mathbb{Q}$ to the outer automorphism group of $\pi_1(\mathbb{P}^1_{\overline{\mathbb{Q}}} \setminus \{0, 1, \infty\})$, is injective. Inspired by Belyi's results, Grothendieck developed his theory of dessin d'enfants as a way to study $G_{\mathbb{Q}}$ "via" $\pi_1(\mathbb{P}^1_{\overline{\mathbb{Q}}} \setminus \{0, 1, \infty\})$. In this seminar, we will take another approach than that of Grothendieck to study the interaction between $G_{\mathbb{Q}}$ and $\pi_1(\mathbb{P}^1_{\overline{\mathbb{Q}}} \setminus \{0, 1, \infty\})$: we will present a result of Petrov [Pet23b] which says, essentially, that Galois representations which comes from geometry can be realized from $\pi_1(\mathbb{P}^1_{\overline{\mathbb{Q}}} \setminus \{0, 1, \infty\})$. To be able to state the theorem more precisely, let F be a number field, let p be a fixed prime, and fix a tangential basepoint 0_v supported at zero. Denote by $\mathbb{Q}_p[\pi_1^{\text{pro-alg}}(\mathbb{P}^1_{\overline{F}} \setminus \{0, 1, \infty\}, 0_v)]$ the regular functions on the pro-algebraic completion of $\pi_1(\mathbb{P}^1_{\overline{F}} \setminus \{0, 1, \infty\}, 0_v)$, and let

$$\mathbb{Q}_p[\pi_1^{\text{pro-alg}}(\mathbb{P}^1_{\overline{F}} \setminus \{0, 1, \infty\}, 0_v)]^{G_F-\text{fin}} \subset \mathbb{Q}_p[\pi_1^{\text{pro-alg}}(\mathbb{P}^1_{\overline{F}} \setminus \{0, 1, \infty\}, 0_v)]$$

be the subspace consisting of elements whose orbits generate a finite-dimensional subspace. Then the main theorem we will study in this seminar is:

Theorem ([Pet23b, Theorem 1.2]). For any separated scheme X of finite type over F and for any $i \in \mathbb{N}$, the semi-simplification of the G_F-representation $H^i(X_{\overline{F}}, \mathbb{Q}_p)$ appears as a subquotient of the space $\mathbb{Q}_p[\pi_1^{\text{pro-alg}}(\mathbb{P}^1_{\overline{F}} \setminus \{0, 1, \infty\}, 0_v)]^{G_F-\text{fin}}$.

This theorem thus states that all the representations which "comes from geometry" can be found as subquotients of $\mathbb{Q}_p[\pi_1^{\text{pro-alg}}(\mathbb{P}^1_{\overline{F}} \setminus \{0, 1, \infty\}, 0_v)]^{G_F-\text{fin}}$. One says that a p-adic representation of G_F is geometric if it is unramified outside a finite set of places containing all those above p, and if the representation is de Rham, a subtle notion from p-adic Hodge theory, at places above p. Any irreducible representation which "comes from geometry", i.e. occurs in the étale cohomology of varieties over F, is geometric. Fontaine–Mazur conjectures that the opposite holds: any irreducible geometric p-adic representation "comes from geometry". A further result of Petrov [Pet23a] shows that all finite-dimensional subquotients of $\mathbb{Q}_p[\pi_1^{\text{pro-alg}}(\mathbb{P}^1_{\overline{F}} \setminus \{0, 1, \infty\}, 0_v)]^{G_F-\text{fin}}$ are geometric in the sense of [FM95]. As an easy consequence of [Pet23b, Theorem 1.2] and the just mentioned result of Petrov, we find a new formulation of the Fontaine–Mazur conjecture:

Theorem ([Pet23b, Conjecture 1.3-1.4])**.** The Fontaine–Mazur conjecture is equivalent to the conjunction of the following two conjectures:

(i) Every irreducible finite-dimensional representation of G_F that appears as a subquotient of $\bar{\mathbb{Q}}_p[\pi_1^{\text{pro-alg}}(\mathbb{P}^1_{\overline{F}}\backslash\{0,1,\infty\},0_v)]^{G_F-\text{fin}}$ comes from geometry, in the sense that it appears as a subquotient of $H^i(X_{\overline{F}},\overline{\mathbb{Q}}_p(j))$ for some variety X and some $i \geq 0$, $j \in \mathbb{Z}$.

(ii) Any irreducible $\overline{\mathbb{Q}}_p$-representation of G_F which is geometric appears as a subquotient of $\bar{\mathbb{Q}}_p[\pi_1^{\text{pro-alg}}(\mathbb{P}^1_{\overline{F}}\setminus\{0,1,\infty\},0_v)]^{G_F-\text{fin}}$.

We will cover the proof of the theorem of [Pet23b, Theorem 1.2] in detail, and as we do this, we will give introduction to the mathematics which goes into the proof in more detail. In the seminar, there will be short introductions to:

- The Fontaine–Mazur conjecture
- p-adic period rings
- Tannakian formalism and the algebraic completion of topological groups
- Tangential basepoints
- Various results around étale cohomology.

Time and Place

- We will meet in person roughly every two or three weeks on *Thursday* in Frankfurt (Robert-Mayer Str. 6-8, Room 711 groß).
- A session will consist of two talks, usually both of 75 minutes in length:

#	Frankfurt
1.	14 : 00 - 15 : 15
☺	coffee break
2.	15 : 45 - 17 : 00

Some weeks, we will have one talk that is 90 minutes in length, and another which is 60 minutes in length. The lengths of the talks are recorded in the schedule. We will always have 30 minutes coffee break between the talks.

Schedule

Introduction, the Fontaine–Mazur conjecture *(Frankfurt, 02.05.2024)*

Talk 0. Introduction (60 minutes). *(Speaker: Magnus)*. Following the introduction of [Pet23b], state Belyi's theorem [Gol12, Theorem 1.1]. Then mention how Belyi's theorem, with some effort, implies Proposition 1.1 of [Pet23b]. Briefly introduce the pro-algebraic completion of $\pi_1(\mathbb{P}^1_{\overline{F}} \setminus \{0,1,\infty\}, 0_v)$, where 0_v is a tangential basepoint supported at 0. Especially describe the ring of regular functions of $\pi_1(\mathbb{P}^1_{\overline{F}} \setminus \{0,1,\infty\}, 0_v)$. Then state the main theorem of the seminar, [Pet23b, Theorem 1.2]. Continue by covering Example 1.6, which gives an easy instance of Theorem 1.2 . Then move on to Example 1.5 of [Pet23b], and lastly, explain briefly how we will prove [Pet23b, Theorem 1.2] essentially following page 3 of [Pet23b], expanding when necessary.

Talk 1. Galois representations coming from geometry and geometric Galois representations (90 minutes). *(Speaker: Ruth)*.
The goal of this talk is to familiarize the audience with the Fontaine–Mazur conjecture. For a smooth projective variety X over a number field F, explain why the Galois representation of G_F on the étale cohomology group $H^i(X_{\overline{F}}, \mathbb{Q}_p)$ is unramified outside a finite set of places S of F, where S includes all the places above p (spread out and use proper and smooth base change [Mil80, VI Theorem 4.1, VI Corollary 2.3]). Mention that at places above p, the action of G_F on $H^i(X_{\overline{F}}, \mathbb{Q}_p)$ is almost never unramified. If however X has good reduction at the place v over p, the Galois representation is *crystalline* at v. Even if X does not have good reduction at v, it is always *de Rham*, a notion we will now define.

Following [Bratner, Section 2.1] introduce the comparison isomorphism between de Rham and Betti cohomology on a smooth projective variety X : this gives rise to a pairing between differential forms and algebraic cycles. Explain that one does not need to tensor with all of $\mathbb{C}$ to obtain a comparison isomorphism, it is enough to tensor with B, the ring of periods. Explain that the theory of p-adic period rings aims to extend this isomorphism to the p-adic situation. Following [Bratner, 2.1, "Étale versus de Rham"], introduce the ring B_{dR}, the relevant comparison theorem and how to recover the de Rham cohomology from the étale cohomology tensored with B_{dR}. Then introduce the crystalline period ring, B_{Cris} [Bratner, 2.1, "Étale versus crystalline"], explain the abstract structures B_{Cris} possesses.

Now for a general topological group G and a topological commutative ring B with a continuous G-action, define what it means to be (F,G)-regular [FO, Definition 2.8], then introduce the notion of a B-admissible representation [FO, Definition 2.12], and then state Theorem 2.13. If time allows, for a p-adic field K, give the classification of the de Rham characters of $G_{\mathbb{Q}_p}$ [Car19, Section 1.1.3, Example 2.2.10].

End the lecture by finally stating the Fontaine–Mazur conjecture [FM95, Conjecture 1], and then follow [Pet23b, Lemma 9.3], which shows that the Fontaine–Mazur conjecture is equivalent to Conjecture 1.3 and Conjecture 1.4 of [Pet23b].

Tannakian formalism and the pro-algebraic completion of π_1 *(Frankfurt, 23.05.2024)*

Talk 2. Tannakian formalism (75 min). *(Speaker: Benjamin)*. Introduce the notion of a neutral Tannakian category [Sza09, Definition 6.5.1]. Give several examples of neutral Tannakian categories [DM12, p. 2.3], make sure to show that the category of representations of an affine group scheme over a field k forms a neutral Tannakian category. Then state [Sza09, Theorem 6.5.3], Give an overview of the proof of the Tannakian reconstruction theorem [Sza09, Theorem 6.5.3], but gloss over details. If time allows, show how properties of the Tannakian category translate into properties of the associated group scheme [DM12, pp. 2.20, 2.21, 2.23].

Talk 3. $\pi_1^{\mathbf{alg}}(X,x)$ and its properties. (75 min). *(Speaker: Amine)*. Using the Tannakian theory developed in the previous talk, introduce the pro-algebraic completion of a topological group G ([Sza09, Example 6.5.14] for discrete G) and introduce also the pro-reductive completion. As an example, calculate the algebraic and reductive completion of $\mathbb{Z}$ and $\hat{\mathbb{Z}}$ [Pet23b, Example 2.3]. Now define the algebraic completion of $\pi_1^{\text{ét}} X,x)$ and use [Sza09, Proposition 6.5.15], together with [Sza09, Remark 6.5.17] to describe the ring of functions of $\pi_1^{\text{alg}}(X,x)$. Then prove [Pet23b, Lemma 2.1] and prove Lemma 2.2 by Tannakian methods by applying [DM12, Proposition 2.21] and "inducing up" a representation. Prove Lemma 2.4 by using the Tannakian perspective as much as possible ([Sza09, Proposition 6.5.15] might be helpful). Then state and prove Lemma 2.6 and 2.7.

Belyi's theorem and tangential basepoints *(Frankfurt, 13.06.2024)*

Talk 4. Belyi's theorem and tangential base-points (75 minutes). *(Speaker: Leonie)*. Prove Belyi's theorem [Gol12, Theorem 1.1, Theorem 2.5]: use the proof of Belyi's theorem involving Vandermonde determinants appearing in Section 2 of [Gol12]. In the remainder of the time, introduce tangential base-points as in [Pet23b, Section 10]. If time allows, it is highly advised to consult [Del89, Section 15] as motivation for the theory, and if possible, present it. State, and prove as much as possible of [Pet23b, Lemma 10.1].

Talk 5. Consequences of Belyi's theorem, direct sums and tensor products (75 minutes). *(Speaker: Marcin)*. Prove Proposition 3.3 of [Pet23b] (state, but do not prove the theorems contained in the external references). Introduce the class $\mathscr{C}_F$ of G_F-representations as on page 9 of [Pet23b], and then prove Corollary 3.4. Then prove Proposition 4.1 [Pet23b], cover the Lemmas as time allows, but make sure to at least sketch the "telescopic property" of the fundamental group of $\mathbb{P}^1 - \{0,1,\infty\}$. Finish by proving Corollary 4.4 and Corollary 4.5 of [Pet23b].

Dual representations, Artin motives and H^1. *(Frankfurt, 20.06.2024)*

Talk 6. Artin motives and H^1 (60 minutes). *(Speaker: Jon)*.
Cover section 5 and section 7 of [Pet23b].

Talk 7. Dual representations and Hodge-Tate characters. *(Speaker: Jakob)*. (90 minutes) The goal of this talk is to cover Section 6 of [Pet23b]. State Proposition 6.1, and explain why it follows from Lemma 6.2, but blackbox the proof of [Pet23a, Proposition 8.5]. Then cover the proof of Lemma 6.2 in as much detail as possible, and expand on the details which are left to the reader. For example, introduce the basics on the Tate elliptic curve and the fact that the associated Galois representation, together with its Ext-class, is as Petrov claims. A possible source for the Tate elliptic curve is [Sil94, pp. V.3–5]. The other main part that goes into the proof of Lemma 6.2 is essentially [FM95, Section 6, Proposition]. If there is time, try to give a (very) high-level overview of Fontaine–Mazur for abelian representations.

Proof of main result and Frobenius eigenvalues *(Frankfurt, 04.07.2024)*

Talk 8. Finishing the proof (60 minutes). *(Speaker: Magnus)*.
Prove [Pet23b, Theorem 1.2]. The proof of [Pet23b, Theorem 1.2] is quite short, and is quite straightforward, if one grants the results from previous lectures and some heavy machinery. If there is time, the lecturer should expand on various details and results contained in the proof.

Talk 9. Frobenius eigenvalues . *(Speaker: Marius)*. Recall Deligne's theory of weights [KW01, Section 1.2] and then prove [Pet23b, Proposition 9.1]. The proof involves results of Lafforgue [Laf02, VII 2)] proving a conjecture of Deligne, as well as Deligne's results on weight-monodromy. For a summary of the results needed from Lafforgue, see [EK12, Section 4.2]. Then prove [Pet23b, Corollary 9.2].

Talk ☺. Dinner in Frankfurt.

Back-up slot *(Frankfurt, 11.07.2024)*

Talk 10. Back-up . *(Speaker: ???)*.
Talk 11. Back-up. *(Speaker: ???)*.

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