Program for the seminar: Purity for the flat cohomology

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1 Introduction

This seminar aims to study a paper by Cesnavicius and Scholze [CS] which proves purity for flat cohomology. The main result is the following

Theorem 1. *For a Noetherian local ring* $(R, \mathfrak{m})$ *that is a complete intersection and a commutative, finite, flat R-group scheme G*

$$H^i_{\mathfrak{m}}(R, G) \cong 0 \text{ for} \begin{cases} i & < \dim(R); \\ i & \leq \dim(R), \text{if } R \text{ is regular and not a field.} \end{cases}$$

This is the flat cohomology version of absolute purity for étale cohomology, proved by Gabber in [Fuj02]. This leads to a purity theorem for Brauer groups

Theorem 2. *Let* $(R, \mathfrak{m})$ *be a Noetherian local ring that is a complete intersection. Let* $U_R := \mathrm{Spec}(R) \backslash \{\mathfrak{m}\}$ *be its punctured spectrum. Then*

$$Br(R) \xrightarrow{\cong} Br(U_R),$$

if $\dim(R) \geq 4$ *or R is regular and* $\dim(R) \geq 2$.

A reference for Theorem 2 for regular rings is [Ces19].

Talk 1: Proof sketch and an introduction to perfectoids

[02.05.2024] State the main theorems as in chapter 1 of [CS]. Guided by section 2.1 of [CS], define perfectoid rings and state their basic properties.

Talk 2: Tilting and the arc topology

[16.05.2024] Define the arc and I-adic arc-topology [CS, Paragraph 2.2.1]. Let $\mathcal{F}$ be a torsion sheaf in the étale topology over A. Prove that the functor on the category of A-algebras A'

$$A' \mapsto R\Gamma_{\text{ét}}(A', \mathcal{F}),$$

satisfies hyperdescent in the arc topology. Prove tilting for perfectoids [CS, Theorem 2.2.7].

If time permits, prove the ind-syntomic generalization of André's lemma [CS, Theorem 2.3.4].

2 Prerequisites

Talk 3: Prime to p aspects of the result

[23.05.2024] In this talk, one studies the cohomological purity for groups G of finite rank n which is prime p. This was shown by Gabber in the 2000s. The proof was written by Fujiwara in his paper [Fuj02]. The paper [CS] gives a proof using perfectoid spaces.

Show the cohomological semi-purity using perfectoid spaces [CS, Theorem 3.1.3]. One explains how to obtain the full cohomological purity from a similar argument [CS, Remark 3.1.4].

Extend the result to general noetherian Rings [CS, Theorem 3.2.4]. Here we replace the dimension of R by its virtual dimension $\mathrm{vdim}(R)$.

Talk 4: Inputs from crystalline and prismatic Dieudonné theory

[06.06.2024] Let S be a perfect $\mathbb{F}_p$-Scheme. State that S-groups which are commutative, finite, and locally free of p-power order are classified by their Dieudonné modules $\mathbb{M}(G)$, which are $W(S)$-algebras with Verschiebung V and Frobenius F. Assume that G is killed by p^n. Prove "the key formula":

$$R\Gamma_Z(S,G) \cong R\Gamma_Z(W_n(S),\mathbb{M}(G))^{V=1} \tag{1}$$

functorially in S,Z and G [CS, Theorem 4.1.8].

Prove [CS, Theorem 4.1.13], which is **Theorem 1** in the equicharacteristic complete intersection case.

If time permits, state the classification in the case of a perfectoid $\mathbb{Z}_p$-algebra A, by replacing crystalline Dieudonné modules by their prismatic counterpart and $W(A)$ by $\mathbb{A}_{\mathrm{inf}}(A)$.

3 Animated rings and their flat cohomology

To prove a perfectoid version of (1), one needs to prove arc descent on both sides, then tackle the problem arc locally, i.e. using valuation rings.

On the left side of the equation, one uses animated rings. Animated rings will be introduced in talk 4, and used to prove excision in talk 6. Excision is necessary to prove arc descent, which will be proved in talk 7.

Talk 5: Introduction to animated rings and cohomology of affine smooth groups

[13.06.2024 Speaker:Tim]

Following subsection 5.1 in [CS] define animated rings and state the deformation theoretic result stated in [CS, Theorem 5.1.13].

Define the flat and étale sites over an animated ring R. Prove theorem 5.2.6 in [CS] which states in particular that the cohomology functor

$$A \mapsto R\Gamma_{\mathrm{fppf}}(A, G)$$

satisfies étale hyperdescent for a given *affine* commutative smooth R-group G.

State the deformation-theoretic statement of [CS, Theorem 5.2.8].

Talk 6: The p-continuity formula

[20.06.2024] Prove the continuity formula [CS, Theorem 5.3.5] comparing fppf cohomology of an animated R-algebra A to the limit of the fppf cohomology of $A/^{\mathbb{L}}p^n$.

Talk 7: Excision for flat cohomology

[27.06.2024 Speaker: Dahli] The aim of this talk is to prove an excision property for flat cohomology of animated rings [CS, Theorem 5.4.4], namely that for a map $A \to A'$ of animated R algebras and for each finitely generated ideal $I \subset \pi_0(A)$ satisfying an adequate "flatness" property, the morphism of flat cohomologies with support in I

$$R\Gamma_I(A, G) \to R\Gamma_I(A', G)$$

is an isomorphism.

Let $(R, \mathfrak{m})$ be a Noetherian local ring and G a commutative finite and flat R-group. Conclude that the local cohomology groups $H^i_{\mathfrak{m}}(R, G), i \in \mathbb{Z}$ do not change after replacing R with its completion $\hat{R}$, see [CS, Corollary 5.4.5.].

4 Proof of the main theorems

We are now ready to prove the main result. In talk 9 we discuss the local version of the result. In talk 10 we will consider the global version and conclude with a proof of the conjecture of Gabber.

Talk 8: The key formula for perfectoid rings

[04.07.2024 Speaker: Amine] We now aim to prove *the key formula* for perfectoids

$$R\Gamma_Z(A, G) \cong R\Gamma_Z(\mathbb{A}_{\mathrm{inf}}(A), \mathbb{M}(G))^{V=1}. \tag{2}$$

Introduce the objects $\mathbb{M}(G)$ as in section 4.2 of [CS]. Check that the right hand side of 2 satisfies hyperdescent for p-complete perfectoid arc hypercovers as stated in [CS, Proposition 4.2.7]. Prove the same for the left-hand side [CS, Theorem 5.5.1].

Prove the perfectoid key formula [CS, Theorem 6.1.1.] and purity of flat cohomology of perfectoid rings [CS, Theorem 6.1.2].

Talk 9: Purity for flat cohomology of local complete intersections

[11.07.2024 Speaker:Morten] The aim of this talk is to prove Theorem 1. To achieve this result one needs the ind-syntomic André lemma [CS, Theorem 2.3.4].

Prove purity for local cohomology of local complete intersection rings R [CS, Theorem 6.2.3]. Prove the sharper result for regular rings [CS, Theorem 6.2.7].

If time permits, prove [CS, Theorem 6.2.4], which is a version of purity for Noetherian local rings, which removes the local complete intersection property by replacing the dimension of R by its virtual dimension $\mathrm{vdim}(R)$.

Talk 10: Global purity and the conjecture of Gabber

[18.07.2024] Purity for local cohomology extends to purity for étale cohomology of schemes. Prove [CS, Theorem 7.1.2]. This leads to purity for gerbes [CS, Theorem 7..1.3]. Conclude by purity for Brauer group [CS, Theorem 7.2.5], as well as purity for the Picard groups [CS, Theorem 7.2.1]. If time permits, extend this to higher degree cohomology [CS, Theorem 7.2.9].

References

[Ces19] Kestutis Cesnavicius. "Purity for the Brauer group". In: *Duke Mathematical Journal* 168.8 (2019), pp. 1461 –1486. DOI: 10.1215/00127094-2018-0057. URL: https://doi.org/10.1215/00127094-2018-0057.

[CS] Kestutis Cesnavicius and Peter Scholze. "Purity for flat cohomology". In: *arXiv preprint arXiv:1912.10932* (2019).

[Fuj02] Kazuhiro Fujiwara. "A proof of the absolute purity conjecture (after Gabber)". In: *Algebraic Geometry 2000, Azumino*. Vol. 36. Mathematical Society of Japan, 2002, pp. 153–184.

Organization

Time: Thursdays from 9:15 to 10:45
Place: Heidelberg, Mathematikon, room: tba

If you have any question contact the email address:
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