GAUS-AG on *Shtukas for reductive groups and global Langlands correspondence after Vincent Lafforgue*

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Summer semester 2024

1. Introduction

Let $\mathbb{F}_q$ be a finite field and let X be a smooth, projective, geometrically irreducible curve over $\mathbb{F}_q$ with function field F and let ℓ be a prime number not dividing q. Let G be a split reductive group over F and let $\widehat{G}$ be its Langlands dual group considered as a split group over $\mathbb{Q}_\ell$ and let $\mathbb{A}_F$ be the ring of adeles over F.

In the Langlands program, one expects a correspondence between two types of objects: cuspidal automorphic representations of $G(\mathbb{A}_F)$ (automorphic side) and continuous irreducible Galois representations $\sigma : \mathrm{Gal}(\overline{F}/F) \to \widehat{G}(\overline{\mathbb{Q}}_\ell)$ unramified outside a finite set of places (Galois side). This correspondence should be compatible with the Satake isomorphism at every unramified place.

For $G = \mathrm{GL}_2$ this correspondence was proved by Drinfeld and for $G = \mathrm{GL}_n$ by L. Lafforgue [Laf02]. In [Laf18], V. Lafforgue established the automorphic to the Galois side of this correspondence for general G. The goal of this seminar is to understand Lafforgue's main ideas in this work.

The structure of the seminar: Since our seminar has rather an unusual style comparing to other GAUS-AGs, in what follows, we would like to describe the structure of our seminar in few words. The first two talks will be conventional 90 minute chalkboard talks. For the remaining ten talks we follow the videos of the workshop at MSRI on the recent progress in Langlands program. Starting with Talk 3, the duration of each session will be approximately 110 minutes. For each session, a moderator will be assigned. For the first 15-20 minutes, the moderator will be responsible for briefly explaining some preliminaries including the tools that the speaker has used without introducing precisely as well as describing an overview for the talk providing a motivation and highlighting the remarkable results discussed there. The goal of this "pre-video" part of the session is to introduce additional materials helping the audience to be more familiar for the technical part of the concept discussed in the talk (see suggestions for the moderator below). After that we will watch the talk. The video of each talk lasts approximately 60 minutes. For the remaining 20-30 minutes of the session, the audience will have an opportunity to discuss and ask questions about the parts of the talk that they did not understand, and together, we will try to clarify them. Hence we suggest the moderator to be familiar with technical details of the talk, getting some assistance from further references listed below if necessary, so that our "post-video" discussion would be more beneficial for everyone.

The seminar will take place on Fridays 9-11am in the room SR8 (tentatively) of Mathematikon in Heidelberg. The Zoom meeting details will be provided later.

2. Talks

Talk 1: Automorphic side (Date: 26.04, Speaker:)
Quickly review what a reductive group is with a focus on GL_n as the basic example [GH19, §1.5]. Define automorphic cuspforms and cuspidal automorphic representations [GH19, §6.5].

Define the local and global Hecke algebras and state that the unramified Hecke algebra is commutative [GH19, §5.5]. Explain Flath's decomposition for the Hecke algebra and for an automorphic representation [GH19, §5.6 and §5.7]. Explain Satake isomorphism for split groups. [GH19, §7.2]. State Weil's uniformization theorem for vector bundles [Gai03] and for split groups [Ric].
Further references: [Yun15, §2-5]

Talk 2: Stacks (Date: 03.05, Speaker:)
Begin the talk by briefly explaining the efficacy of the definition of a stack using the example of rank n vector bundles [Hei10, §1]. After defining stacks [Hei10, Def. 1.1], provide examples like Bun_n, BG, $[X/G]$ etc. Next motivate the notion of an *atlas*, and consequently, geometric properties of a stack, with the help of BG [Hei10, §1.2]. Define an algebraic stack [Hei10, Def. 1.10] and also discuss briefly what it means for it to be étale, smooth etc. [Hei10, Def'n.2.1]. Then define a *Deligne-Mumford* stack and an *Artin* stack [Gom99, Def. 2.20,2.22]. Conclude the talk by giving a sketch of [Gom99, Ex.2.24] showing Bun_n is an Artin stack.

Talk 3: Moduli of shtukas I (Date: 10.05, Session Moderator: Oğuz Gezmiş)
Video: Talk 3
Notes: Weinstein (part I)
Suggestions for the moderator: This talk introduces one of the main objects of this seminar, the shtukas. It mainly uses the martial from the last two talks but it would be nice to restate Satake isomorphism in the language of this talk and quickly review the theory of weights for reductive groups [Mil15, §22].
Further references: [Yun15, §6], [YZ17, §5], [DV23, §2]

Talk 4: Moduli of shtukas II (Date: 17.05, Session Moderator:)
Video: Talk 4
Notes: Weinstein (part II)
Suggestions for the moderator: This talk is about more general moduli stacks of shtukas and their cohomologies. It would be nice to state basic facts about cohomology of modular curves and Deligne's construction of the Galois representation associated with eigenforms [Sai06].
Further references: [Yun15, §9], [Dri87, §1]. [Yun15, §6]

Talk 5: Geometric Satake (Date: 24.05, Session Moderator:)
Video: Talk 5
Notes: Cass
Suggestions for the moderator: This talk focuses on the Beilinson-Drinfeld Grassmannian and (a version of) geometric Statke equivalence between representations of (powers of) $\widehat{G}$ and preserve sheaves on the affine Grassmannian. The moderator can review some basic definitions of homological algebra (symmetric monoidal category, semi-simple category, etc) as they see fit and also recall the definition of the Hecke stack [Ric] and its relation to moduli stack of shtukas (see here).
Further references: [BR18, §4], [Gross], [Yun15, §11] [Zhu17, §A], [Laf14, §3]

Talk 6: Cohomology of moduli of shtukas (Date: 31.05, Session Moderator:)
Video: Talk 6
Notes: Morel

Suggestions for the moderator: This talk discusses the local models for moduli stack of shtukas and the construction of the functor $\mathcal{H}_I$ whose generic stalk would be the functor H_I of Lafforgue from representations of (powers of) $\widehat{G}$ to Hecke-Galois modules (this will be discussed in talk 8). The main ingredient is the Geometric Satake from last talk. Other than reviewing material from previous talks, it would be nice to review local and global Hecke correspondences for moduli of shtukas [Yun15, §5].
Further references: [Xue20b, §2.1], [Laf14, §3]

Talk 7: Finiteness of cohomology of moduli of shtukas (Date: 07.06, Session Moderator:)
Video: Talk 7
Notes: Xue
Suggestions for the moderator: The main theme of this talk is to discuss cohomology of the sheaves introduced in Talk 6. It mainly uses many ingredients from the previous talk as well as apply Drinfeld's lemma to prove certain results. Hence it would be useful to recall these notions while giving an overview of the talk.
Further references: [Xue20b, §2]

Talk 8: Specialization to the diagonal (Part I) (Date: 14.06, Session Moderator:)
Video: Talk 8
Notes: Yun (part I)
Suggestions for the moderator: The main players of this talk are certain Ind-constructible sheaves and their specializations. It is basically self-contained, except some parts used from Talk 7. Therefore one can briefly recall the Ind-constructrible sheaf defined in Talk 7 as well as some of its properties at the beginning of the session.
Further references: [Xue20, §1.1], [Laf14, §1.1-1.2]

Talk 9: Specialization to the diagonal (Part II) (Date: 21.06, Session Moderator:)
Video: Talk 9
Notes: Yun (part II)
Suggestions for the moderator: Since this is a continuation of the previous talk, it would be useful to recall main parts of Talk 8 in addition to providing an overview for the topics discussed in the present talk
Further references: [Xue20, §1.2], [Laf14, §1.1-1.2]

Talk 10: Drinfeld's Lemma (Date: 28.06, Session Moderator:)
Video: Talk 10
Notes: Chan
Suggestion for the moderator: Here the speaker shows that a geometric Galois group of a product of curves factors through the product of Galois group. The theory of finite dimensional l-adic representations and some basic commutative algebra is used therein. Descent type results for objects equipped with (partial) Frobenius is also a key ingredient.
The moderator could discuss beforehand the "Reminder" discussed in the talk (also given in the notes), for instance by recalling the objects used therein.
Further references: [Dri89, §6], [Xue20b, §3.2-3.3]

Talk 11: Moduli of representations and global Langlands parametrization (Part I) (Date: 05.07, Session Moderator:)
Video: Talk 11

Notes: Caraiani (part I)
Suggestions for the moderator: In the first of a two part series of the talks, the speaker states a crucial result of V. Lafforgue, decomposing the functor H_I in terms of global Langlands parameters. Next, the speaker discusses the key ingredients of the proof as well as a motivation for the necessity of using excursion operators. Later on the speaker explicitly constructs an excursion operator.
In the preliminary discussion, the moderator could remind the definition of the functors H_I and briefly say how $H_\emptyset(\mathbb{1})$ relates to cuspidal automorphic forms.
Further references: [Xue20b, §3.6], [Laf14, §2]

Talk 12: Moduli of representations and global Langlands parametrization (Part II) (Date: 12.07, Session Moderator:)
Video: Talk 12
Notes: Caraiani (part II)
Suggestions for the moderator: The speaker delineates on two of the key ingredients/ideas involved in the proving the main theorem of V. Lafforgue stated in previous talk, namely quasi coherent sheaves on stacks and pseudo representations. A result relating the functor H_I to quasi coherent sheaves on the quotient stack $\mathrm{Hom}(\Gamma, M)/\widehat{G}$ is stated (à la Drinfeld-Zhu). Then pseudo representations are related to $E[FFG]-$algebra homomorphisms.
One could try to present beforehand, some details about the construction of the "framed $M-$valued representation variety" $\mathrm{Hom}(\Gamma, M)$.
Further references: [Xue20b, §4], [Laf14, §5]

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