GAUS AG on Chromatic Homotopy Theory

Program by Timo Richarz and Georg Tamme

Summer term 2024

The goal of the seminar is to provide an introduction to chromatic homotopy theory and its relation to the moduli stack of 1-dimensional formal groups. The final talks follow the recent work of Barthel, Schlank, Stapleton and Weinstein [Bar+24] that computes the rationalized homotopy groups of the $K(n, p)$-local spheres for all $n \in \mathbb{Z}_{\geq 0}$ and all prime numbers p, thereby confirming the so called chromatic splitting and vanishing conjectures rationally.

Structure of talks, some prerequisites and comments

The major part of the seminar provides background material in chromatic homotopy theory following [Lurc], see also [BB20] for a recent survey on the subject. The final part study the recent manuscript [Bar+24]. The first part of the seminar gives results and proofs in some detail, but it will become more and more sketchy the further we progress. The final part gives an overview of the proof of [Bar+24, Theorems A & B], but certainly not all details. Every participant is invited to be curious, to raise questions and to make comments. Provided enough interest, we can always arrange for extra talks, discussion sessions or invite external experts on the subject to help us with digesting the material.

Generally speaking, the program assumes familiarity with ∞-categories and some higher algebra. For the first part, a background in algebraic topology is certainly helpful, though not strictly necessary. For the final part, we assume some familiarity with perfectoid spaces or condensed mathematics. Here we do not aim for completeness. Still everybody should be able to get "something out of each talk". The talks labelled by $*$ are independent of the other talks and suitable for advanced master's students. These are **Talks 2, 3 & 9**.

An aim of the seminar is also to get a feeling for how arguments are done in chromatic homotopy theory. As a general guideline, each talk should explain at least one "typical" argument in detail. The organizers leave the choice of argument to each speaker and encourage everyone to discuss this with other participants, ideally before the talk. In case you are interested in participating but not sure what talk to give, please feel free to contact the organizers.

Time and place

The seminar takes place in a hybrid format jointly organized by Darmstadt and Mainz.

- Tuesdays, 14:00 – 15:30 during the summer term 2024.
- Start date: April 16, end date: July 16
- The last session on July 16 is a double session with 2 talks between 14:00 – 18:00. Afterwards there will be a joint dinner. Details will be announced later.
- The seminar will take place in Darmstadt, Room S215 401 and Mainz, Room 04-432.
- Zoom meeting ID: 612 2072 7363, Password: Largest six digit prime number.

Talks

Leitfaden (Apr 16)

This is an overview talk (20–30 minutes) given by one of the organizers explaining the interrelation of talks and the structure of the seminar.

Talk 1 - Spectra & (co)homology theories (Apr 16, only 60 minutes due to the Leitfaden)

Give an overview on some results from algebraic topology including the following key words: spaces, spectra, (co)homology theories and Brown's representability theorem. Important examples: Eilenberg–MacLane spectra, (periodic) complex K-theory KU. This talk should be given by someone with a background in algebraic topology. References: [Ada95, Part III], [Lurb, §1.4]. Please have also a look at the overview in [Lurc, Lecture 1] since the next couple of talks follows the lecture notes.

Talk 2* - Lazard's theorem (Apr 23)

Follow [Lurc, Lectures 2 & 3] and prove Lazard's theorem as time permits (Theorem 4 in the notes). Note that the term "formal group" in the notes refers to 1-*dimensional* formal groups, which is the only type of formal groups considered in our seminar.

Talk 3* - Complex-oriented cohomology theories (Apr 30)

Follow [Lurc, Lecture 4]. In particular, explain how to pass from complex-oriented multiplicative cohomology theories to formal group laws (Proposition 10 in the notes), see also the overview in [Lurc, Lecture 1].

Talk 4 - Complex bordism (May 7)

Follow [Lurc, Lectures 5 & 6]. Introduce the complex bordism spectrum MU and prove that complex orientations on commutative ring spectra are classified by ring spectrum maps out of MU (Theorem 8 in Lecture 6).

Talk 5 - Milnor–Quillen theorem on MU (May 14)

Follow [Lurc, Lecture 7]. State Theorem 1 in the notes, i.e., Quillen's result that $\pi_*(\mathrm{MU})$ is canonically isomorphic to the Lazard ring (the existence of an abstract isomorphism is due to Milnor). Explain the proof on the level of homology in detail. Sketch the remaining ingredients as time permits, see [Lurc, Lectures 8–10] and, in particular, Theorem 4 in Lecture 10.

Talk 6 - The moduli stack of formal groups (May 21)

Follow [Lurc, Lectures 10–14]. Introduce the moduli stack $\mathcal{M}_{\mathrm{FG}}$ of formal groups (Definition 2 in Lecture 11) and its variant $\mathcal{M}^s_{\mathrm{FG}}$ with strict isomorphisms (Definition 5 in Lecture 10). Please feel free to formulate the definition directly in terms of a moduli problem of (coordinatizable) formal groups. Introduce heights of formal groups (Lecture 12) and the height stratification on $\mathcal{M}_{\mathrm{FG}}$ (Lecture 13). End by discussing some structural properties following Lecture 13, in particular, state Theorem 10 on the classification of formal groups (its proof can be found in Lecture 14).

Talk 7 - Flat modules over $\mathcal{M}_{\mathrm{FG}}$ (May 28)

Follow [Lurc, Lectures 15 & 16]. Explain that flat modules on $\mathcal{M}_{\mathrm{FG}}$ induce homology theories on spaces and proof Landweber's exact functor theorem, which gives a criterion for a module over $\mathcal{M}_{\mathrm{FG}}$ to be flat.

Talk 8 - Even periodic cohomology theories (Jun 4)

Follow [Lurc, Lectures 17 & 18]. The aim is to formulate and prove Proposition 11 in Lecture 18. The Hopkins–Miller theorem [Rez98] is an ∞-categorical enrichment of this result, see also [Lura, Theorem 0.0.8].

Talk 9* - Lubin–Tate theory (Jun 11)

Follow [Lurc, Lectures 19 & 21] (skip Lecture 20). Show that the moduli stack $\mathcal{M}_{\mathrm{FG}}^{n}$ of formal groups of height n is the fppf quotient of $\mathrm{Spec}(\bar{\mathbb{F}}_p)$ by the Morava stabilizer group (Proposition 1 in Lecture 19). Analyze the Morava stabilizer group following Lecture 19. Discuss Lubin–Tate deformation theory following Lecture 21, in particular, Remark 8 is important. Finish by giving some homework to the participants: In the next talk, we need some generalities on Bousfield localization as in [Lurc, Lecture 20], see also [MNN17, §§2–3] for a nice treatment.

Talk 10 – Morava E-theory & Morava K-theory (Jun 18)

Follow [Lurc, Lectures 22 & 23]. Introduce Morava E- & K-theory. State the smash product theorem (Theorem 1 in Lecture 22) without proof. Explain the proof of Proposition 2 in Lecture 23 in detail. In particular, Proposition 5 in Lecture 23 is important. (The generalities on Bousfield localization are given as homework to the participants at the end of Lecture 9.)

Talk 11 – Localizing subcategories of p-local spectra (Jun 25)

Follow [Lurc, Lectures 25 & 26]. The aim is to prove the thick subcategory theorem (Theorem 8 in Lecture 26). This rests on the nilpotence theorem from Lecture 25. If time permits elaborate on Remark 9 in Lecture 26. As a consequence we are able to determine the Balmer spectrum of the category of spectra, see [BB20, §2.1] and the references therein. Please note that this is the last talk following Lurie's lectures. There are several nice results we have to skip, e.g., the chromatic convergence theorem (Theorem 8 of Lecture 29), the proof of the smash product theorem in Lecture 31, or the study of the monochromatic layers in Lectures 34 & 35. For further reading, we refer to the survey [BB20] and the references cited therein.

Talk 12 – Chromatic splitting & vanishing conjectures (Jul 2)

This talk is more advanced than all previous ones. The reference is [Bar+24, §2] with the aim to prove Proposition 2.6.3 (Theorem B implies Theorem A). For this, the speaker needs to connect the material, especially in §§2.3–2.4, with things we have seen in previous talks. In particular, it seems necessary to elaborate on Equations (2.3.1) and (2.3.3), especially Devinatz–Hopkin's pro-Galois extension $L_{K(n)}S^0 \to E_n$ and a rudiment understanding of the spectral sequence is crucial. Then, state the Conjectures 2.4.2 & 2.4.4 mentioned in the title. Depending on time the speaker can also state the weak splitting conjecture (see [BB20, Conjecture 2.20] for details and also the relevance of the conjectures). Let us remark that the material in §2.5 seems not strictly necessary for the proof of Proposition 2.6.3. It is up to the speaker to skip this. Instead it might be more relevant to sketch the proof of Lemma 3.8.4.

Talk 13 – The isomorphism between the two towers (Jul 9)

The main source for this talk is [SW13], see also [SW20, §23.3]. This talk assumes familiarity with perfectoid spaces as we will have no time to introduce the theory. The aim is to explain the proof of [Bar+24, Theorem 3.9.1], i.e., the isomorphism between the Lubin–Tate tower at ∞-level and the Drinfeld tower at ∞-level.

Talks 14 & 15 – Proof of Theorem B (Jul 16, double session)

These talks should be prepared together. The aim is to say something meaningful about the proof of Theorem B following [Bar+24]. One possibility would be that the first speaker focusses on Theorem 3.9.3 and, for example, choose one of the situations. This is based on the machinery of integral p-adic Hodge theory after Bhatt–Morrow–Scholze. The second speaker could then reap the benefits of this work and finish the proof, including the existence of the $\mathbb{G}_n$-invariant splitting $A = W \oplus A^c$ and the computations of the continuous cohomology groups in (3.9.4).

References

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