SUPERCONNECTIONS, THETA SERIES, AND PERIOD DOMAINS

GAUS AG (SoSe 2023): Reading seminar on Garcia's paper [Gar18].
Organizers: Jan Bruinier, Yingkun Li, Riccardo Zuffetti.

Place: Hybrid seminar at TU Darmstadt, S2|15 244, streamed in Zoom.
Format: Each meeting is made of one talk of *90 minutes*.
Time: We meet every week on Wednesdays from 13:15 to 14:45, starting from 10th May 2023. If many people are interested, we could split some talks and begin some weeks earlier.

In the celebrated article [HZ76], Hirzebruch and Zagier proved that the intersection numbers of certain curves on Hilbert modular surfaces are Fourier coefficients of some modular forms. This was the first instance of the nowadays classical problem of considering some generating series of geometrical objects (such as intersection numbers, cohomology classes, rational classes etc) and ask whether they have modular properties.

During a long collaboration in the eighties, Kudla and Millson provided a method to construct Siegel modular forms whose Fourier coefficients are cohomology classes of *special cycles* on *orthogonal Shimura varieties.* Since Hilbert modular curves are orthogonal Shimura varieties of dimension 2, and the curves considered in [HZ76] are examples of special cycles, the results of Kudla and Millson may be considered as a generalization of [HZ76].

To provide more details, we now introduce some notation. Let X be a smooth orthogonal Shimura variety arising from an even lattice L of signature $(n, 2)$. For simplicity, we may assume L to be unimodular and $n > 2$. The n-dimensional quasi-projective variety X admits a natural Kähler form ω, arising from a $\mathrm{SO}(L\otimes\mathbb{R})$-invariant form on some Hermitian symmetric domain of type IV, in particular X is also a Kähler manifold.

The special cycles of X are suitable sums of orthogonal Shimura subvarieties of X, and are parametrized by half-integral positive semi-definite $g \times g$ matrices, for any $g \leq \dim X$. We denote by Λ_g the set of these matrices, and by $Z(T)$ the special cycle arising from $T \in \Lambda_g$. The latter has codimension $\mathrm{rk}(T)$ in X and induces a cohomology class $[Z(T)]$ in the singular cohomology group $H^{2\,\mathrm{rk}(T)}(X, \mathbb{C})$.

For every $g \leq \dim X$, Kudla and Millson constructed a theta function $\Theta(\tau, z, \varphi_{\mathrm{KM}})$ associated to a certain Schwartz function φ_{KM} on $(L \otimes \mathbb{R})^g$, where the variable τ lies in the genus g Siegel upper half space $\mathbb{H}_g$, and $z \in X$. This function behaves with respect to τ as a (non-holomorphic) Siegel modular form of weight $1 + \dim X/2$, and with respect to z as a closed $2g$-differential form on X. They also proved [KM90] that the *cohomology class* (as a differential form) of this theta function is a holomorphic Siegel modular form with Fourier expansion

$$[\Theta(\tau, z, \varphi_{\mathrm{KM}})] = \sum_{T\in\Lambda_g} [Z(T)] \wedge [-\omega]^{g-\mathrm{rk}(T)} e^{2\pi i\,\mathrm{tr}(T\tau)}. \tag{1}$$

This is the so-called *generating series* of the cohomology classes of special cycles in codimension g.

In the recent paper [Gar18], Luis Garcia illustrates a new way to construct and generalize the theta function $\Theta(\tau, z, \varphi_{\mathrm{KM}})$. He also provides a shorter proof of the modularity of the generating series (1).

The goal of this reading seminar is to learn together the main tools of [Gar18], such as superconnections on period domains, as well as the new proof of the modularity of the generating series. Eventually, following the wording of Garcia–Sankaran [GS19], we will apply these new methods to construct Green forms for special cycles and use them to prove the so-called arithmetic Siegel-Weil formula over $\mathbb{R}$.

1. Talks

What does $\star$, $\star\star$, and $\star\star\star$ mean?

$\star$: Suitable for Ph.D. students without much background in algebraic geometry.
$\star\star$: Suitable for Ph.D. students and postdocs; usually requires background in algebraic geometry or knowledge of almost all previous talks.
$\star\star\star$: Suitable for ambitious Ph.D. students and postdocs, as well as for Professors. Requires solid background in algebraic geometry and/or the willingness to engage with the material in significant depth.

Mathematics is made together.

The goal of the Gaus-AG is not only to learn advances in current research, but also to get to know each other, discuss and participate actively. We strongly encourage you to ask the organizers and your colleagues about any problem arising in the preparation of the talks. Questions about general state of art and curiosities on the topic of the seminar are more than welcome.

Talk 1: Some classical results of Kudla and Millson. $\star$ (10.5 Darmstadt) [Fabian Scherf]

Recall how to construct (connected) orthogonal Shimura varieties X_L from an even indefinite lattice L of signature $(n, 2)$. For simplicity, restrict to the case of unimodular lattices. You may also assume X_L to be smooth. For every symmetric half-integral positive semi-definite $g \times g$ matrix, recall how to construct the associated special cycle of codimension g on X_L; see [Zuf22, Section 7, pp. 19-20]. Recall that special cycles induce linear functionals on the space of compactly supported closed forms on X_L, hence they induce cohomology classes in $H^{2g}(X_L, \mathbb{C})$ by Poincaré duality; see [BT82, Chapter 5, p. 50] for the general theory. Recall classical (scalar valued) Siegel modular forms of general genus g and their Fourier expansions [Bru+08, Part 3, Sections 2-4]. State (without proof) that the generating series of cohomology classes of codimension g special cycles on X_L is a Siegel modular form of genus g and weight $1 + \dim X_L/2$; see [Kud04, Theorem 3.1].

Caveat: There are several equivalent ways to define special cycles. We suggest to stick with $X_L = \Gamma\backslash\mathcal{D}$, where $\mathcal{D}$ is the Hermitian symmetric domain associated with $\mathrm{SO}(n, 2)$. The speaker is welcome to ask the organizers for any doubt or clarification.

Talk 2: Superconnections. $\star$ (17.5 Darmstadt) [Mingkuan Zhang]

Give an overview of the content of [Gar18, Appendix A], with help of the details provided in [BGV92, Sections 1.3-1.6] and [Qui85]. Define superconnections on super vector bundles, the associated Chern character forms and illustrate their basic properties [Gar18, Section A.1]. Illustrate the additional properties of Chern characters of *holomorphic* vector bundles on complex manifolds as in [Gar18, Section A.2]. Define Koszul complexes and their metrics arising from Hermitian metrics of holomorphic vector bundles [Gar18, Section A.3].

Talk 3: Period domains and invariant forms. ⋆⋆ (24.5 Darmstadt) [Jiaming Chen]

The goal of this talk is to give an overview of [Gar18, Section 2]. Recall construction and basic properties of period domains as in [Gar18, Sections 2.1-2.4], further details can be found in [CMSP17, Sections 4.4 and 15.1]. Introduce the invariant form φ as in [Gar18, Section 2.5] and illustrate its properties [Gar18, Proposition 2.3]. Prove that φ is rapidly decreasing [Gar18, Proposition 4.1].

Talk 4: Relations with the Kudla–Millson forms. ⋆⋆ (31.5 Darmstadt) [Gabriele Bogo]

The goal of this talk is to give an overview of [Gar18, Section 3]. Restrict to the case of Hermitian symmetric domains associated with $\mathrm{SO}(n,2)$ [Gar18, Sections 3.1-3.2]. Compute the invariant form φ introduced in the previous talk in this case [Gar18, Sections 3.3-3.4]. Following [Gar18, Sections 3.5-3.7], compare the form φ with the Kudla–Millson form φ_{KM}; details on the latter can be found in [Kud97, Section 7]. Illustrate the proof of [Gar18, Theorem 3.2].

Talk 5: Weil Representation and Modularity. ⋆⋆ (14.6 Darmstadt) [Christina Röhrig]

Recall the Weil representation ω of the metaplectic group over the adeles on Schwartz spaces, illustrate the behavior of φ with respect to ω (see [Pra93, sections 1-2], [Gel93, seciton 1], [Ada07, sections 1-4]). Use this representation and φ to define a theta series $\theta(\tau;\mu+L^r)$ (see [Gar18, Definition 5.1]). Discuss relation to the theta series defined in talk 1 (also consult [Kud03, section 1]). Introduce the form ν in [GS19, Definition 2.2.4], give some explicit formulas for $\mathrm{O}(p,2)$ in [GS19, section 2.3], and discuss its basic properties in [GS19, section 2.4]. Prove [Gar18, Lemmas 4.3] and [GS19, Lemma 2.5.4] regarding the action by the maximal compact subgroup on φ and ν through the Weil representation. Prove Lemma 4.5 in [Gar18], and the modularity of the cohomology class of the theta series in [Gar18, Theorem 5.2].

Talk 6 : Green Forms for Special Cycles I. ⋆⋆⋆ (21.6 Darmstadt) [Riccardo Zuffetti]

Define Green forms following [BGS94, section 1.1]. Discuss the current defined by φ on $\mathbb{D}$. Prove Proposition 4.8 in [Gar18] and deduce the Fourier coefficients of the theta series $\theta(\tau;\mu+L^r)$ defined in talk 5 (see section 5.3 in [Gar18]). Introduce the currents $\xi^\circ(u)$ and $\mathfrak{g}^\circ(u)$ for regular u in section 2.1 of [GS19]. Prove their basic properties and relation to φ in [GS19, Proposition 2.1.7]. If time permits, discuss the definition of $\mathfrak{g}^\circ$ in the irregular case.

Talk 7 : Green Forms for Special Cycles II. ⋆⋆⋆ (28.6 Darmstadt) [Yingkun Li]

Define the currents $\mathfrak{g}^\circ(\mathbf{v})$ and their star products of [GS19, section 2.7]. Prove Theorem 2.7.1. Recall special cycles on orthogonal Shimura varieties [GS19, sections 4.1-4.2], and use the Green form $\mathfrak{g}^\circ$ to define Green forms for these special cycles in Definition 4.3.5.

In the process prove Proposition 4.3.4. Discuss star products of these currents and prove Theorem 4.5.1.

Talk 8: Green forms and the Siegel–Weil formula. ⋆⋆⋆ (5.7 Darmstadt) [Jan Bruinier]

Recall the Siegel Eisenstein series for double cover of the symplectic group and the Siegel-Weil formula [GS19, section 5.1]. State properties of its Fourier coefficients E_T in Lemma 5.2.3 and Proposition 5.2.12. State and prove Theorem 5.3.1.

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