Program GAUS AG Prismatization

Summer semester 2023

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Time The seminar is now planned for Monday 14:00 – 15:30.

Place The seminar will take place in hybrid form. Talks can be given in person on blackboard and filmed with a camera (much preferred) or via Zoom. At the end of the semester all participants will be invited to come to Darmstadt, to listen to two talks at once, and to have a jolly good dinner together.

Mathematics Our main references are [Bhatt] and [BL1].

More precisely, we want to read [Bhatt]. The goal is the geometrization of p-adic cohomology theories with their additional structures. The final goal is a geometrization of prismatic cohomology together with its additional structures including realization functors to étale cohomology, crystalline cohomology, and de Rham cohomology.

The essential idea is the following. Given a "cohomology theory" defined over a suitable base ring Σ [1], construct a functor

$$\mathcal{S}\mathrm{ch}/\Sigma \to \mathcal{S}\mathrm{tacks}/\Sigma, \qquad S \mapsto B_S,$$

a ring-valued[2] stack $\mathcal{H}$, and a map of stacks $\mathcal{H} \to B$ such that if $f\colon X \to S$ is an "admissible map" of schemes (or formal schemes, or whatever), to which the cohomology theory can applied, we can make the following construction. Denote by $f^{\mathcal{H}}\colon X^{\mathcal{H}} \to B_S$ the map defined on points by

$$X^{\mathcal{H}}(\mathrm{Spec}(R) \to B_S) := X(\mathcal{H}(R)).$$

Then the $\mathcal{H}$-cohomology (with its additional structures) of $f\colon X \to S$ is given by $Rf^{\mathcal{H}}_*\mathscr{O}_{X^{\mathcal{H}}} \in D(B_S)$[3]. In particular, B_S should be a stack encoding the extra structure existing on the cohomology.

Let us make this rather vague principle more explicit for de Rham cohomology, endowed with its Hodge filtration, for morphisms of p-adic schemes. Let V be a p-complete ring[4] and let X/V be a smooth qcqs p-adic formal

[1] Since we are mainly interested in p-adic cohomology theories, for us this base ring will be often $\mathbb{Z}_p$. But for instance if you are interested in de Rham cohomology in characteristic 0, then one could choose $\mathbb{Q}$ as a base ring.

[2] with a suitably modern definition of "ring"

[3] For a stack $\mathcal{X}$, $D(\mathcal{X})$ denotes a suitable derived category of $\mathscr{O}_{\mathcal{X}}$-modules. If $\mathcal{X}$ is a scheme X, then $D(\mathcal{X}) = D_{\mathrm{qc}}(X)$, i.e. the derived category of $\mathscr{O}_X$-modules with quasi-coherent cohomology.

[4] with the correct definition of p-completeness

scheme. We want to apply the above principle to geometrize de Rham cohomology $R\Gamma(V, \Omega^\bullet_{X/V})$ considered as a complex together with its Hodge filtration. With the notation above, we set $B_V := \mathbb{A}^1/\mathbb{G}_m \times V$. Then we will see that an object in $D(B_V)$ is an object of the filtered derived category of V. In [Bhatt, 2.5] there is defined a stack $\mathcal{H} = \mathbb{G}_a^{dR,+}$ in commutative rings and a map $\mathbb{G}_a^{dR,+} \to \mathbb{A}^1/\mathbb{G}_m$ such that if one sets as above for a ring R

$$X^{dR,+}(R) := X(\mathbb{G}_a^{dR,+}(R)),$$

the following holds. Denoting by $\pi^{dR,+}$ the map $X^{dR,+} \to B_V$, then its derived direct image $R\pi_*^{dR,+}\mathscr{O}_{X^{dR,+}}$ identifies with the Hodge filtered de Rham complex $R\Gamma(X, \Omega^\bullet_{X/V})$. Moreover, vector bundles on $X^{dR,+}$ give rise to the natural coefficient systems for Hodge-filtered de Rham cohomology: they correspond to triples $(\mathscr{E}, \nabla, \mathrm{Fil})$ consisting of a filtered vector bundle $(\mathscr{E}, \mathrm{Fil})$ on X together with a flat connection $\nabla\colon \mathscr{E} \to \Omega^1_{X/V} \otimes \mathscr{E}$ that satisfies Griffiths transversality with respect to the filtration and such that ∇ has nilpotent p-curvature modulo p.

Plan for the seminar

We follow [Bhatt] and construct these geometrizations for more and more interesting cohomology theories $\mathcal{H}$ (de Rham $\to$ crystalline $\to$ prismatic $\to$ syntomic). After background talks on rings and stacks, the plan is to proceed for the cohomology theories as follows.

(1) Recall the cohomology theory and its additional structures.

(2) Encode these additional structures into an algebraic stack ("Linear Algebra via stacks"), i.e. define the correct functor $S \mapsto B_S$ (with the above notation).

(3) Attach to a map $f\colon X \to S$, to which we want to apply the cohomology theory[5], a map $f^{\mathcal{H}}\colon X^{\mathcal{H}} \to B_S$ and show/motivate that $Rf_*\mathscr{O}_{X^{\mathcal{H}}}$ can be identified with the cohomology of f.

(4) Study vector bundles/perfect complexes on $X^{\mathcal{H}}$ as natural coefficient systems for the cohomology.

Moreover, we relate the different cohomology theories via their geometrizations.

Prerequisites

(1) Participants should have some ideas about some basic notions about ∞-categories (as worked out by Lurie). For a nice very concise introduction see for instance [Sch, p. 13ff].

(2) Participants should also be familiar with some homological algebra including the notion of derived categories, t-structures, and perfect complexes.

[5] Here we will not hesitate to make simplifying assumptions that help to understand the essential idea without getting bogged down in generalities, e.g., S might be the spectrum of a perfect field – at least for de Rham and crystalline cohomology.

A short[6] introduction to most of these topics is given in [Kha].

Notation All categories are ∞-categories. Classical categories are called 1-categories. Stacks are derived stacks, i.e. étale sheaves with values in the category of anima (= spaces = ∞-groupoids).

Talks

There are two "double talks" (Talk 2+3 and Talk 5+6), each taking two sessions (= 180 minutes). They should be given by two speakers who decide themselves how to divide the material (or by one ambitious speaker...). The last three talks will be given in workshop form on July 10 starting at 10:00.

Talk 1: **Andreas Gieringer: Animated rings**, April 17

(1) Explain the animation of a projectively generated 1-category $\mathcal{C}$ [Mao, App. A][7], explaining the case of animation of sets (leading to the category of *anima* or *spaces*) and of modules over a given ring R. In particular mention that the latter is the connective derived category $\mathcal{D}_{\geq 0}(R)$.

(2) Now animate the 1-category of commutative rings to obtain the category of animated rings [BL1, App. A]. Explain n-truncation of animated rings. Identify 0-truncated rings with classical (= static) rings.

(3) Explain how every quasi-ideal gives rise to a 1-truncated animated ring [Dri2, 3.4], in particular describe the cone of a quasi-ideal $d\colon I \to C$. Give some examples, e.g. the animated ring corresponding to the quasi-ideal $R \xrightarrow{f} R$, where R is a classical ring and the arrow is multiplication with some element $f \in R$.

(4) Make explicit the groupoid of 1-morphisms from a ring A to the cone of a quasi-ideal [Dri1, 1.3.5].

Talk 2+3: **Georg Tamme+Tom Bachmann: Filtrations and endomorphisms via stacks**, April 24 and May 8

Main references: [Bhatt, Section 2.2.1 and 2.2.2], [BL1, App. D], and [Mou].

(1) Explain quotient stacks of a scheme by a functor of groups, see for instance [Kha, Section 4.4, in particular Theorem 4.28][8].

(2) Explain the notion of a graded and a filtered derived category. Explain completeness, canonical and stupid filtration, mention its symmetric monoidal structure without defining precisely what a symmetric monoidal structure is[9], and explain the standard and the Beilinson t-structure.

[6] compared to Lurie's work

[7] In the reference the animation of an arbitrary projective generated n-category is explained. We will not need this generality.

[8] In the reference the quotient by an arbitrary group stack is explained. We do not need this level of generality.

[9] See for instance [Sch, p. 19ff] or the references given therein for the interested participant.

(3) Introduce the quotient stack $\mathbb{A}^1/\mathbb{G}_m$, explain that it classifies generalized Cartier divisors, see [Bhatt, 2.2.5] or [KhRy, 3.2].
(4) Formulate and prove [Bhatt, 2.2.6 + 2.2.8].
(5) Explain $\widehat{\mathbb{G}_a}$ and $\widehat{\mathbf{V}(E)}$ for a vector bundle E.
(6) Explain [Bhatt, 2.2.12 + 2.2.13] without going too much into detail although it would be nice to see, where "characteristic zero" is needed.
(7) If time permits, explain [Bhatt, 2.2.14 – 2.2.16].
(8) In any case, explain [Bhatt, 2.2.17].

Talk 4: **Rizacan Çiloğlu: Warm-Up: De Rham cohomology in characteristik** 0, May 15

This talk covers parts of [Bhatt, 2.1+2.3]

(1) Recall the algebraic de Rham complex of a scheme map $X \to S$ and mention that it is the initial object in the category of strictly commutative differential graded $f^{-1}\mathscr{O}_S$-algebras over X. Define the Hodge complex.
(2) Define de Rham cohomology, its Hodge filtration, and Poincaré duality in the form of [Bhatt, 2.1.4][10].
(3) Explain the general principle of transmutation [Bhatt, 2.3.8]. This is a central point for the rest of the seminar!
(4) Define the ring stack $\mathbb{G}_a^{dR,+} \to \mathbb{A}^1/\mathbb{G}_m$ and show that it captures algebraic de Rham cohomology in characteristic 0 via transmutation. Give as much details of the proof of [Bhatt, 2.3.6] as possible.

Talk 5+6: **Patrick Bieker + Lorenzo Mantovani: $\mathbb{G}_a^{\#}$ and $\mathbb{G}_a^{dR}$**, May 22 and June 5

These are two talks covering [Bhatt, 2.4+2.5.1+2.5.2+2.6].

(1) Define $\mathbb{G}_a^{\#}$ and $\mathbf{V}(E)^{\#}$ and describe their representations [Bhatt, 2.4.1 – 2.4.6].
(2) Recall very briefly the ring scheme of (p-typical) Witt vectors as in [Bhatt, beginning of 2.6] (see [BouACIX, §1] and [Gro, chap. 1] for more background).
(3) The group scheme $\mathbb{G}_a^{\#}$ via Witt vectors [Bhatt, 2.6.1 – 2.6.6].
(4) Flat cohomology of $\mathbb{G}_a^{\#}$ [Bhatt, 2.4.7+2.4.8].
(5) The ring stacks $\mathbb{G}_a^{dR,+}$, $\mathbb{G}_a^{dR}$, and $\mathbb{G}_a^{\mathrm{Hodge}}$ [Bhatt, 2.5.1+2.5.2].
(6) The Witt vector model and the homology of $\mathbb{G}_a^{dR}$ [Bhatt, 2.6.8+2.6.11].

Talk 7: **Marcin Lara: Hodge filtered de Rham cohomology of p-adic formal schemes**, June 12

Main reference: [Bhatt, 2.5.3ff]

(1) Recall the principle of transmutation and define the Hodge filtered de Rham stack and the Hodge stack [Bhatt, 2.5.3].
(2) Formulate and prove [Bhatt, 2.5.6].
(3) Express $(X/V)^{dR}$ as a quotient stack [Bhatt, 2.5.7].

[10] Do not explain Cartier duality or anything particular to characteristic p here!

(4) Describe vector bundles on $(X/V)^{dR,+}$ and $(X/V)^{\mathrm{Hodge}}$ [Bhatt, 2.5.8+2.5.9].
(5) Show the "crystalline miracle" [Bhatt, 2.5.10] (note that we have already seen in Talk 6 that $\mathbb{G}_a^{dR}(R)$ has a functorial V/p-structure by [Bhatt, Footnote 17].

Talk 8: **Christopher Lang: The conjugate filtration in characteristic** p, June 19

Main reference: [Bhatt, 2.7+2.8].

(1) Reminder on the conjugate filtration and the Cartier isomorphism [Bhatt, 2.1.5].
(2) Explain how to consider $(X/k)^{dR}$ as a gerbe over $X^{(1)}$ [Bhatt, 2.7.1] (recall that for a sheaf of groups G a G-gerbe is a sheaf of 1-groupoids, locally equivalent to a sheaf of the form BG, see e.g. [EHKV, 3.1]).
(3) Conjugate filtration via $(X/k)^{dR} \to X^{(1)}$ [Bhatt, 2.7.2].
(4) Conjugate filtration via transmutation [Bhatt, 2.7.7 – 2.7.9].
(5) The Witt vector model for $\mathbb{G}_a^{dR,c}$ [Bhatt, 2.7.11 + 2.7.12].
(6) The Witt vector model for $\mathbb{G}_a^{dR,+}$ [Bhatt, 2.8.1].
(7) Glueing Hodge and conjugate filtration [Bhatt, 2.8.2 – 2.8.4, 2.8.6, 2.8.7].

This talk is rather long. Suggestions to shorten the talk: Focus on the $X^C \to C$, more precisely, explain (1) and (5), formulate conjugate filtration via transmutation using $\mathbb{G}_a^{dR,c,W}$ and do not prove [Bhatt, 2.7.9]. State (6). Then explain the ring stack $\mathbb{G}_a^C$ in detail [Bhatt, 2.8.3] and prove as much of [Bhatt, 2.8.6] as possible. Mention $\bar{C}$ [Bhatt, 2.8.7] briefly as an outlook.

Talk 9: **Thibaud van den Hove: Prismatization in characteristic** p, June 26

Main topic is [Bhatt, 3.1] but we start a little bit more generally.

(1) Define the notion of a Cartier-Witt divisor on a p-nilpotent ring [Bhatt, 5.1.2 – 5.1.5].
(2) Explain the prismatization of a bounded p-adic formal scheme [Bhatt, 5.1.6 – 5.1.7 + 5.1.9 + 5.1.11].
(3) Specialize to schemes in characteristic p [Bhatt, 5.1.12].
(4) Explain [Bhatt, 3.1.1], in particular the relation to crystallization. For this give the definition of crystalline cohomology as in [Bhatt, 2.5.12].

Talk 10: **Manuel Blickle: Nygaard filtered prismatization over a perfect field**, July 3

Main reference: [Bhatt, 3.2+3.3]. The main point is [Bhatt, 3.3.5].

(1) Recall the Nygaard filtration for smooth schemes over a perfect field [Bhatt, 3.2.1 – 3.2.4].
(2) Define and describe $k^{\mathcal{N}}$, the ring stack $\mathbb{G}_a^{\mathcal{N}}$ [Bhatt, 3.3.1 – 3.3.2], and Nygaard filtered prismatization by transmutation [Bhatt, 3.3.3 – 3.3.6].

Talk 11: **Can Yaylali: Prismatic gauges over a perfect field**, July 10

This talk essentially covers [Bhatt, 3.4].

(1) Define and describe (prismatic) gauges over a perfect field k [Bhatt, 3.4.1 – 3.4.3, 3.4.5 – 3.4.10].
(2) Dsecribe vector bundles on $k^{\mathcal{N}}$ [Bhatt, 3.4.11, 3.4.12, 3.4.15].

Talk 12: **Torsten Wedhorn: Syntomification of smooth schemes over perfect fields**, July 10

This talks essentially covers [Bhatt, 4.1+4.3].
(1) Define X^{Syn} directly via glueing [Bhatt, 4.1.1 – 4.1.5] and via transmutation [Bhatt, 4.1.6]
(2) Dsecribe vector bundles on k^{Syn} as crystals [Bhatt, 4.3.1].
(3) Explain Breuil-Kisin twists [Bhatt, 4.3.3 – 4.3.6].

Talk 13: **Anton Güthge: F-Gauges and syntomic cohomology**, July 10

This talks essentially covers [Bhatt, 4.2+4.4].
(1) Define F-gauges and prove that $\mathcal{H}_{\mathrm{Syn}}(X)$ is perfect if X is smooth and proper over k [Bhatt, 4.2.1 – 4.2.4].
(2) Describe F-gauges on k [Bhatt, 4.2.8].
(3) Show that syntomic cohomology of the structure sheaf if p-adic étale cohomology [Bhatt, 4.2.5 – 4.2.6] and define more generally syntomic cohomology or arbitrary weights and some of its properties [Bhatt, 4.4.1 – 4.4.5].
(4) Explain and dwell with relish on [Bhatt, 4.4.11].
(5) If time permits, describe the syntomic cohomology in weight 1 [Bhatt, 4.4.7 – 4.4.8].

References

[Bhatt] B. Bhatt, *Prismatic F-Gauges*,
https://www.math.ias.edu/ bhatt/teaching/mat549f22/lectures.pdf

[BL1] B. Bhatt, J. Lurie, *Absolute Prismatic Cohomology*, arXiv:2201.06120

[BouACIX] N. Bourbaki, *Algèbre commutative*, Chap. IX.

[Dri1] V. Drinfeld, *Prismatization*, arXiv:2005.04746v5.

[Dri2] V. Drinfeld, *On a notion of ring groupoid*, arXiv:2104.07090.

[EHKV] D. Edidin, B. Hassett, A. Kresch, A. Vistoli, *Brauer groups and quotient stacks*, American Journal of Mathematics **123** (2001),761–777.

[Gro] A. Grothendieck, *Groupes de Barsotti-Tate et cristaux de Dieudonné*, les presses de l'université de Montréal (1974).

[Kha] A. Khan, *A modern introduction to algebraic stacks*, https://www.preschema.com/lecture-notes/2022-stacks/stacksncts.pdf

[KhRy] A. Khan, D. Rydh, *Virtual Cartier divisors and blow-ups*, arXiv:1802.05702v2.

[Mao] Z. Mao, *Revisiting derived crystalline cohomology*, arXiv:2107.02921.

[Mou] T. Moulinos, *The geometry of filtrations*, Bulletin of the London Mathematical Society, **53** (2021), 1486–1499.

[Sch] P. Scholze, *Six-functor formalism*, https://people.mpim-bonn.mpg.de/scholze/SixFunctors.pdf