AG-Seminar on *Plectic Stark-Heegner points*

Organizers: Prof. Dr. Böckle, Dr. Gräf
Wintersemester 21/22
Tuesdays 9-11, SR 8 & Online

Overview

The famous theorem of Mordell-Weil states that the set of rational points $A(F)$ of an elliptic curve A, defined over a number field F, is a finitely generated abelian group. The rank of its free part, the so called algebraic rank $r_{\mathrm{alg}}(A/F)$, turned out to be a very subtle invariant. The fundamental discovery of Birch and Swinnerton-Dyer was that, at least computationally, one can use local, and easy to compute, data about the elliptic curve to reconstruct its algebraic rank and other global arithmetic information. More precisely, the local data can be used to build the complex L-function $L(A, s)$ attached to A. The famous Birch and Swinnerton-Dyer (BSD) conjecture then states that the Taylor expansion of $L(A, s)$ around $s = 1$ is of the form

$$L(A,s) = \frac{\#\text{Ш}_{A/F}\Omega_{A/F}R_{A/F}\prod_{\mathfrak{p}} c_{\mathfrak{p}}}{\sqrt{|\Delta_F|}(\#A(F)_{\mathrm{tor}})^2} \cdot (s-1)^{r_{\mathrm{alg}}(A/F)} + [\text{higher order terms}].$$

Here, $\text{Ш}_{A/F}$ denotes the Tate-Shafarevich group of A over F, $R_{A/F}$ is the regulator, $\Omega_{A/F}$ is the real period of A over F and the $c_{\mathfrak{p}}$ are the Tamagawa numbers. In particular, the order of vanishing of $L(A, s)$ at $s = 1$ is conjectured to be precisely $r_{\mathrm{alg}}(A/F)$.
Apart from statistical methods, almost all progress towards the BSD conjecture and surrounding questions has been limited to elliptic curves of small rank, which arguably is due to the fact that the only known general approach to prove that $r_{\mathrm{alg}}(A/F) \geq r$ is to construct r linearly independent points. Thus, it is a natural question to ask for a *systematic* strategy for finding linearly independent points on elliptic curves. In practice, it often turns out to be useful to consider an auxiliary quadratic extension E/F. Then one can even formulate the following Kolyvagin-type problem.

Problem: Suppose $r_{\mathrm{alg}}(A/E) \geq r$. Construct an element $w \in \wedge^r A(E)$ such that

$$w \text{ non-torsion} \Longrightarrow r_{\mathrm{alg}}(A/E) = r.$$

For $r = 1$, this problem is adressed by the construction of Heegner and Stark-Heegner points – at least conjecturally: Stark-Heegner points are (contrary to regular Heegner points) only conjecturally algebraic. However, a wealth of numerical and theoretical evidence suggests that they are in fact algebraic, see [Dar], [Gre], [GMS] and [GMM].
The goal of this seminar is to study a generalization of the p-adic construction of Stark-Heegner points due to Fornea-Gehrmann and Fornea-Guitart-Masdeu in [FG] and [FGM], which tackles the problem above for $r \geq 2$ and is strongly inspired by the plectic conjectures of Nekovář and Scholl, see [Nek] and [NS]. A simplified version of their construction is given as follows: For a certain set S of p-adic prime ideals of F with $|S| = r$, they construct a canonical element P in a local group $\widehat{A}(E_S)$. This local group admits a so called determinant map $\det\colon \wedge^r A(E) \to \widehat{A}(E_S)$. If $r_{\mathrm{alg}}(A/E) \geq r$, [FG] conjecture that the element P is algebraic, in the sense that $P = \det(w)$ for some $w \in \wedge^r A(E)$, and moreover, that if P is non-zero, one has $r_{\mathrm{alg}}(A/E) = r$.

In the seminar we will primarily study the preprint [FG], that is, construct plectic p-adic invariants and plectic Stark-Heegner points. We will also see how these relate to higher derivatives of anticyclotomic p-adic L-functions and learn about their conjectural properties. Additionally, we will study the computational method and evidence presented in the preprint [FGM].

Talks

1) Introduction and overview (Date: 02.11.2021 Speaker: Peter Gräf)
This talk will give an overview over the goals and motivation of the seminar. The remaining talks will be distributed thereafter.

2) The plectic philosophy (Date: 09.11.2021 Speaker: Marius Leonhardt)
References: [Nek], [NS]
This talk will give us a glipse into the plectic world, i.e., into the motivation and the central ideas surrounding Nekovář's and Scholl's plectic conjectures following [Nek] and [NS]. The material from this talk will not be needed in the remainder of the seminar, but it should serve as a guiding motivation.

3) Automorphic representations I (Date: 16.11.2021)
References: [Tho, Lectures 09/10/2013 – 09/24/2013]
This will be the first of two talks covering the essentials on automorphic forms and representations. In this talk, the focus will be on the theory for GL_2 following [Tho, Lectures 09/10/2013 – 09/24/2013]. In particular, the material from [Tho, Definition 4] onwards should be covered, leading up to the central [Tho, Definition 17] and the examples thereafter. Also mention [Tho, Theorem 30]. Briefly introduce the L-function attached to an automorphic representation, see [Bor], and explain the notion of modularity of an elliptic curve. Finally, define the Steinberg representation, see [Tho, Proposition 19 & Example 25] and mention [Tho, Example 26]. Possible additional references for this talk are [Bum] and [BJ].

4) Automorphic representations II (Date: 23.11.2021)
References: [Tho, Lecture 10/08/2013], [Voi], [FG, Section 2]
In this talk, we expand the notions from the previous talk to reductive groups over general bases. For this, we follow [Tho, Lecture 10/08/2013] up to [Tho, Remark 44]. The main examples we are interested in arise from quaternion algebras. Discuss local and global quaternion algebras, ramification and the classification by ramification sets. An excellent reference is [Voi]. It might be instructive to discuss [Tho, Example 15]. State the Jacquet-Langlands correspondence, see [GJ, Theorem 8.3]. End the talk by introducing the setup and notations in [FG, Section 2], which we will use throughout the seminar.

5) Harmonic cochains and the Steinberg representation (Date: 30.11.2021)
References: [FG, Section 3], [DT]
The first aim of this talk is to find a cohomology class attached to our elliptic curve in the cohomology of arithmetic groups. To do this, present the material in [FG, Section 3.1], in particular [FG, Proposition 3.1]. The next aim is to show that we can also find a

corresponding cohomology class with values in harmonic cocycles. Introduce the Bruhat-Tits tree and harmonic cochains following [FG, Section 3.2]. Of particular importance are the exact sequences (10) and (12) in [FG]. More details can for example be found in [DT]. Introduce the notation in [FG, Section 3.3] and sketch the proof of the crucial [FG, Proposition 3.6].

6) Plectic p-adic invariants (Date: 07.12.2021)
References: [FG, Sections 4.1 & 4.2], [DT, Section 1]
In the first part of the talk, introduce the p-adic integration theory as in [FG, Section 4.1]. Introduce the p-adic upper half plane $\mathcal{H}_p$ and explain its relation to the Bruhat-Tits tree, see [DT, Section 1]. Continue with the construction of the map $(\Psi_S^{\diamond})^*$ as in [FG, Section 4.1.1]. After introducing the (twisted) fundamental class, define plectic p-adic invariants following [FG, Section 4.2]. End the talk by explaining the Tate-uniformization of elliptic curves with split multiplicative reduction and how it can be used to produce a so called *plectic point*; for more details on Tate-uniformization see [FvdP, Section 5.1].

7) Plectic Stark-Heegner points (Date: 14.12.2021)
References: [FG, Sections 4.3 – 4.5]
The aim of this talk is to refine the constructions from the previous talk to obtain plectic Stark-Heegner points. The central ingredient are certain extensions of the Steinberg representation constructed by Breuil and Spieß. Introduce these extensions following [FG, Section 4.3] and explain the exactness of the sequence in [Spi1, Lemma 3.11]. Then introduce (naive) divisors on $\mathcal{H}_p(E_\mathfrak{p})$ and prove [FG, Lemma 4.7], see also [BG, Lemma 6.8]. State the conjectural lifting theorem [FG, Theorem 4.9] and use it to define plectic Stark-Heegner points. Briefly explain the relation to the plectic points from the previous talk. End the talk by sketching the proof of [FG, Theorem 4.9] in the case where the field F is totally real following [FG, Section 4.5]. The proof is quite technical and relies on the equality of arithmetic and automorphic $\mathcal{L}$-invariants proved in [GR, Theorem 4.1] and [Spi2, Theorem 3.7]. Provide as many details as time permits.

8) The p-adic Gross-Zagier formula (Date: 11.01.2022)
References: [FG, Section 5]
The aim of this talk is to construct an anticyclotomic p-adic L-function attached to the elliptic curve A/F and the quadratic extension E/F and to prove a p-adic Gross-Zagier formula relating its higher derivatives to plectic p-adic invariants. After discussing the background material at the beginning of [FG, Section 5.1], study characters of the torus following [FG, Section 5.2] and construct the p-adic L-function, see [FG, Definition 5.7]. The second part of the talk focuses on special values of this p-adic L-function. Introduce the notation from the beginning of [FG, Section 5.4] and define the period integral. Define the local linear functionals and state Waldsburger's formula, see also [YZZ]. State [FG, Theorem 5.10] and sketch the idea of the proof. Finally, present the material in [FG, Section 5.5] and explain in particular [FG, Theorem 5.13].

9) Computing plectic p-adic invariants (Date: 18.01.2022)
References: [FG, Section 1.4], [FGM]
The first part of this talk should be devoted to stating various conjectures on plectic p-adic

invariants and plectic Stark-Heegner points, in particular regarding their algebraicity. For this, follow [FG, Section 1.4] and [FGM, Section 1.2]. The aim of the second part of the talk is to explain the numerical evidence and the method of computation presented in [FGM].

References

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