# GAUS AG: Six functor formalism and Poincaré duality

Sommersemester 2023

AG Schmidt (Arithmetic Homotopy Theory), Heidelberg

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**Time and place: Thursday 09:30** at **SR 8** in Mathematikon, Heidelberg (if sufficiently many people are interested, online participation might be possible)
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**Literature:** The main references are [Sch23] and [Zav23].

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## Introduction

The aim of this GAUS AG is to understand (the motivation for and) the nature of a 6 functor formalism (6FF), and in particular its interplay with (Poincaré/Verdier/...) duality. Let $X$ be a geometric object (topological space, scheme, ...) and consider the category $\mathrm{Ab}(X)$ of abelian sheaves (or étale torsion sheaves or ...) on $X$. A fundamental invariant of an abelian sheaf on $X$ is its collection of cohomology groups, conveniently packaged into the derived functor $R\Gamma(X,-)\colon D(X) \to D(\mathrm{Ab})$, where $D(X) := D(\mathrm{Ab}(X))$.

For every map $f\colon X \to Y$, we have an exact pullback functor $f^*\colon D(Y) \to D(X)$, which has a right adjoint $f_*$, i.e. $\boxed{f^* \dashv f_*}$. Moreover, the category $D(X)$ has a tensor product $\otimes$ and an internal hom $\underline{\mathrm{Hom}}$ with the usual Hom-tensor-adjunction $\boxed{- \otimes B \dashv \underline{\mathrm{Hom}}(B,-)}$ for all $B$. Finally, at least for certain maps $f\colon X \to Y$, we also have a "proper pushforward" functor $f_!\colon D(X) \to D(Y)$ and its right adjoint $f^!$, i.e. $\boxed{f_! \dashv f^!}$. The six functors $(f^*, f_*, \otimes, \underline{\mathrm{Hom}}, f_!, f^!)$ satisfy many compatibilities, which often carry geometric meaning (proper base change, Künneth formula, projection formula, ...). A good definition of 6FF should encompass all these compatibilities.

Mann [Man22] and Scholze [Sch23] propose the following way of packing all those requirements into one definition. Let $C$ be a category (of geometric objects, e.g. schemes ...) and $E$ be a class of morphisms in $C$ (those for which $f_!$ should exist).

**Definition 1** *A* 3FF *is a lax symmetric monoidal functor*

$$D\colon \mathrm{Corr}(C,E) \to \mathrm{Cat}_\infty\,.$$

*A* 6FF *is a 3FF for which the functors $f^*$, $-\otimes A$ and $f_!$ (which are encoded in $D$) admit right adjoints.*

Before diving into this abstract framework, the GAUS AG starts with the example of sheaves of abelian groups on topological spaces, motivating the geometric meaning of the six functors. Our next goal is to understand Definition 1, i.e. to learn some background about $\infty$-categories, to construct the $\infty$-category of correspondences $\mathrm{Corr}(C,E)$, to unpack the definition, and to show that it really does encode the compatibilities we want. Moreover, we will see that — as is often done in practice — we can build a 6FF from two classes of morphisms $I, P \subset E$ of "open immersions" (for which $f_! \dashv f^*$) and "proper maps" (for which $f_! = f_*$) if any $f \in E$ admits a "compactification" $f = \overline{f}j$ with $\overline{f} \in P$ and $j \in I$.

The final goal of this GAUS AG is to understand Poincaré duality (PD) in this setup. The duality is already expressed by the adjunction $f_! \dashv f^!$, but it remains to "compute" $f^!$. Scholze defines cohomologically smooth maps $f \in E$ as those for which one has a formula for $f^!$. Zavyalov [Zav23] then interprets PD as (1) identifying when (in a geometric situation) smooth implies cohomologically smooth, and (2) calculating $f^!$ as $f^*(-) \otimes \omega_f$ for smooth $f$ with some invertible $\omega_f \in D(X)$. He shows that cohomologically smooth is equivalent to the existence of a trace-cycle theory, which reduces (1) to the construction of one trace map ($+\varepsilon$), e.g. for the projective line. He also introduces the concept of a theory of first Chern classes, which reduces (2) to cohomology computations for $\mathbb{P}^1$.

## Talks

After an introductory first talk, talks 2 and 3 give an overview of the 6FF for sheaves on topological spaces. For these talks, we particularly encourage master students to volunteer. Talks 4-7 explain Scholze's 6FF and Definition 1. The remaining talks (8 and 9, but maybe more) deal with PD.

Note that this program is quite flexible. If you need more than 90 minutes for your talk, that's fine, but please let us know. If the audience's interest shifts to a particular aspect, we can incorporate that. That is why the dates are provisional and the topics, in particular of the later talks, are tentative. If you have any questions when preparing your talk, please don't hesitate to contact us.

**Talk 1: Introduction** (20/04)
Speaker: Marius Leonhardt
Main reference: [Sch23, Lecture I]
Introduce the category of abelian sheaves $\mathrm{Ab}(X)$ on a topological space $X$ together with the functors $-\otimes- \dashv \underline{\mathrm{Hom}}(-,-)$ and $f^* \dashv f_*$ (for a continuous map $f\colon X \to Y$). Motivate a 6FF using this example, in particular giving geometric meaning to the involved functors and stressing how Proper Base Change, Künneth formula, ... fit into the picture. Give an overview of this GAUS AG and distribute the talks.

**Talk 2: Direct image with compact support** (27/04)
Main references: [GM03, III.8, 1–14], [Mat11, §1-2]
Define the higher direct image with compact support $f_!$. Compute the stalks of $f_!F$ and show that the functor $f_!$ is exact when restricted to *soft* sheaves (i.e. "soft sheaves are adapted to the functor $f_!$"). In the remainder of the talk, introduce briefly the derived category of complexes of abelian sheaves that are bounded from the left. Present the derived functors of the five functors that have been discussed before. Introduce as much homological algebra as you see fit.

**Talk 3: Inverse image with compact support** (04/05)
Main references: [GM03, III.8, 15–28], [Mat11, §3-5]
Introduce the notion of dimension for locally compact topological spaces. Prove the main theorem [GM03, 16. Theorem] that the derived direct image with compact support $\mathrm{R}f_!$ admits a right adjoint functor $f^!\colon \mathrm{D}^+(\mathrm{Ab}(Y)) \to \mathrm{D}^+(\mathrm{Ab}(X))$ for a continuous map $f\colon X \to Y$ between locally compact and finite-dimensional spaces. For this, use without proof the formal criterion [GM03, 19. Theorem] about the representability of a functor. In the end, introduce the dualising complex and deduce Verdier duality [GM03, 27. Corollary].

**Presumably no talk** (11/05)

**No talk (holiday)** (18/05)

**Talk 4: 6FF: $\infty$-categorical background** (25/05)
Main reference: [Sch23, Lecture II]
Give the list of desiderata of a 6FF. Motivate and explain the words $(n,m)$-category, 2-category and $\infty$-category. Give some background on simplicial sets and define an $\infty$-category. State a list of properties of $\infty$-categories and especially symmetric monoidal structures on them and state Definition 1 with some explanation of $\mathrm{Corr}(C,E)$.

**Talk 5: Symmetric monoidal $\infty$-categories and 6FF** (01/06)
Main reference: [Sch23, Lecture III]
Define $\mathrm{Corr}(C,E)$ properly. Define and explain symmetric monoidal $\infty$-categories and how to write down (lax symmetric monoidal) functors into $\mathrm{Cat}_\infty$. Explain how to equip $\mathrm{Corr}(C,E)$ with a symmetric monoidal structure and why Definition 1 encodes e.g. the projection formula (and many more desirable compatibilities).

**No talk (holiday)** (08/06)

**Talk 6: 6FF: construction via compactification** (15/06)
Main reference: [Sch23, Lecture IV]

Explain the assumptions on $I$ and $P$ and sketch how to extend a "2FF" to a 3FF if we know $f_!$ for $f \in I \cup P$ and that any $f \in E$ has a decomposition $f = \overline{f}j$ with $\overline{f} \in P$ and $j \in I$. In particular, explain why the independence of $f_! := \overline{f}_! j_!$ from the choice of the decomposition is easy to see in this framework. Introduce bisimplicial sets and explain their role in the construction.

**Presumably no talk** (22/06)

**Talk 7: Poincaré duality** (29/06)

Main reference: [Sch23, Lecture V]

Define cohomologically smooth $f \in E$. Characterise them [Sch23, Theorem 5.5] and sketch as much of the proof as possible. In particular, construct the 2-category $\mathrm{LZ}_{\mathcal{D}}$ associated to a 3FF $\mathcal{D}$. Also explain the example of PD for topological spaces.

**Presumably no talk** (06/07)

**Talk 8: Cohomologically smooth/proper morphisms and duality** (13/07)

Main references: [Sch23, Lecture VI], [Zav23, §2.3]

For $f\colon X \to Y$ the morphism into the final object, define $f$-proper and $f$-smooth objects $A \in D(X)$. State the duality results and define cohomologically proper and cohomologically étale $f$.

**Presumably no talk** (20/07)

**Talk 9: Abstract Poincaré duality** (27/07)

Main reference: [Zav23, §3]

State "formal" Poincaré duality. Define a trace theory and a trace-cycle theory for $f$ and state Poincaré duality for such $f$. Reduce the question of "$\forall f$ : smooth $\Rightarrow$ cohomologically smooth" to the existence of a trace-cycle theory for the projective line.

**Alternative:** [Zav23, §4-5] Give an overview of the geometric arguments that give a formula for the dualizing object in terms of the relative tangent bundle. Introduce Chern classes and show that they can be used to both construct the trace map and to trivialize the dualizing object.

**Closing event: Hike** (27/07)

Traditionally, we go for a little hike through the hills of Heidelberg and have dinner together after the last talk. Details will be announced in due time.

# Literatur


[GM03] Sergei I. Gelfand and Yuri I. Manin. *Methods of homological algebra.* Springer Monographs in Mathematics. Springer-Verlag, Berlin, second edition, 2003.

[Man22] Lucas Mann. A $p$-Adic 6-Functor Formalism in Rigid-Analytic Geometry. *arXiv e-prints*, page arXiv:2206.02022, June 2022.

[Mat11] Akhil Mathew. Verdier Duality. Expository Notes (version dated July 29, 2011), available at `https://math.uchicago.edu/~amathew/verd.pdf`, 2011.

[Sch23] P. Scholze. Six-functor formalisms. available at `https://people.mpim-bonn.mpg.de/scholze/SixFunctors.pdf`, 2023.

[Zav23] Bogdan Zavyalov. Poincaré Duality Revisited. *arXiv e-prints*, page arXiv:2301.03821, January 2023.