

VECTORIAL DRINFELD MODULAR FORMS OVER TATE ALGEBRAS

1. INTRODUCTION

The goal of this seminar series is to gain more insight on the structure of vectorial Drinfeld modular forms. The first appearance of these objects dates back to the work of Pellarin [Pel12] where he obtained special values of Dirichlet L -functions in the positive characteristic case by using such forms. In 2018, a further study of these forms, which we will be more interested in, are introduced by Pellarin and Perkins [PP18]. More precisely, they determined the algebraic structure of these interesting functions and further investigated their relation with Drinfeld modular forms for some congruence subgroups as well as Hecke eigenforms. This direction of research developed recently by Pellarin [Pel21] into a more general setting.

Before saying further, let us be more explicit and briefly mention what we mean by vectorial Drinfeld modular forms. We start with setting up some notation. Let p be a prime number and q be a positive power of p . We let \mathbb{F}_q be the finite field with q elements. Consider a variable θ over \mathbb{F}_q and set $A := \mathbb{F}_q[\theta]$. We let K be the fraction field of A and K_∞ be the completion of K at the infinite place, which is indeed the ring of formal Laurent series in θ^{-1} with coefficients in \mathbb{F}_q . We finally set \mathbb{C}_∞ to be the completion of a fixed algebraic closure of K_∞ and define the Drinfeld upper half plane $\Omega := \mathbb{C}_\infty \setminus K_\infty$ which is a connected rigid analytic space (see [Gos80b, §1]).

A Drinfeld modular form $f : \Omega \rightarrow \mathbb{C}_\infty$ of weight $k \in \mathbb{Z}$ and type $m \in \mathbb{Z}/(q-1)\mathbb{Z}$ for a congruence subgroup Γ of $\mathrm{GL}_2(A)$ is a rigid analytic function satisfying

$$f\left(\frac{az+b}{cz+d}\right) = (cz+d)^k \det(\gamma)^{-m} f(z), \quad z \in \Omega, \quad \gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma,$$

and having also a certain growth condition. As any reader who has a basic knowledge on (elliptic) modular forms can predict, there are several similarities between the theory of classical modular forms and Drinfeld modular forms. They will be explained in detail during the second talk of our seminar series.

Let \mathbb{T} be the Tate algebra, the ring of power series in a variable t with coefficients in \mathbb{C}_∞ , and as a function of the same variable, converging in the closed unit disk. Let r be a positive integer. We will be interested in functions $g : \Omega \rightarrow \mathbb{T}^r$, *vectorial modular forms*, satisfying certain automorphy and growth conditions. Since each coordinate g_i of g is \mathbb{T} -valued, for certain points $\xi \in \mathbb{C}_\infty$ such as the roots of unity, the evaluation map ev_ξ sending g_i to $g_i(\xi)$ is well-defined. More importantly, in some particular cases, the composite function $\mathrm{ev}_\xi \circ g_i : \Omega \rightarrow \mathbb{C}_\infty$ becomes a Drinfeld modular form of some level. The seminar series mainly focuses on obtaining several properties of vectorial modular forms when $r = 2$ as well as their applications to special values of L -functions and Drinfeld modular forms. In addition, we will define Hecke operators acting on the space of vectorial modular forms, and, via applying hyperderivatives and evaluating at roots of unity, construct Hecke eigenforms for certain congruence subgroups of $\mathrm{GL}_2(A)$.

In what follows, we briefly explain the details of each talk. To simplify, in several places, the terminology *vectorial Drinfeld modular form(s)* will be abbreviated as $\mathrm{VDMF}(s)$.

2. TALKS

Talk 1: Background on Drinfeld modules**Speaker: Luisa Pauline Boneberger****Date: April 25th, 2023**

The first talk aims to give the audience some necessary background for the rest of the seminar. First, set up the notation as explained above and define Drinfeld modules over \mathbb{C}_∞ . Discuss the exponential function corresponding to a Drinfeld module as well as its periods [Gos96, §3, 4], [Gek88, §2]. To illustrate the content further, explain the details for the Carlitz module and define its fundamental period [EGP14, §2,4]. The correspondence between rank r A -lattices and Drinfeld modules of rank r (Drinfeld Uniformization Theorem) must be stated, and if time permits, a sketch of its proof would be given [BP20, §2.4]. The talk should be finalized with a discussion on Anderson generating functions. Its particular properties, due to Pellarin [Pel08, §4.2], should be explained and a sketch of their proof will be given [EGP14, Prop. 3.2, Prop. 6.2].

Talk 2: Background on Drinfeld modular forms**Speaker: Theresa Häberle****Date: May 2nd, 2023**

Our goal in the second talk is to give an exposition of known results in the theory of Drinfeld modular forms, mainly, for the full modular group $\mathrm{GL}_2(A)$ [Gek88], [Gos80a]. To be more precise, as a starting point, some background on rigid analytic (holomorphic) functions on the Drinfeld upper half plane Ω should be given [Rev92, §1–2] (see also [Gos92, §2–5], [Ste97] and [FvdP04, §2.2]). Later on, weak Drinfeld modular forms as well as Drinfeld modular forms (for $\mathrm{GL}_2(A)$) and their Fourier expansions shall be discussed and the condition of holomorphy at infinity must be explained [Gek88, §5]. Our main objects for this talk are going to be illustrated via providing several examples such as Eisenstein series, coefficient forms, h -function of Gekeler, or more generally, Poincaré series [Gek88, (5.9), (5.10), (5.11)]. The results on the \mathbb{C}_∞ -algebra structure of Drinfeld modular forms must be stated [Gek88, 5.12, 5.13]. If time permits, Hecke operators shall be briefly introduced and the notion of Hecke eigenvalues and Hecke eigenforms will be explained [Gek88, §7].

Talk 3: Introduction to weak vectorial Drinfeld modular forms**Speaker: Paola Francesca Chilla****Date: May 9th, 2023**

Our goal in this talk is to introduce weak vectorial Drinfeld modular forms which will have a crucial role to determine special values of Goss L -functions. We need to emphasize that VDMFs given in [Pel12, Def. 12] are indeed seen as weak VDMFs in [PP18, Def. 3.4]. Throughout the seminar, we will borrow this terminology and call them weak VDMFs. Our main goal for the talk is to analyze the \mathbb{C}_∞ -vector spaces of a certain subclass of weak VDMFs studied in [Pel12]. The talk will start with basic definitions. Later on we prove [Pel12, Lem. 13] which indeed implies that one dimensional weak VDMFs corresponding to the trivial representation $\mathbf{1}$ are nothing but weak Drinfeld modular forms tensored with \mathbb{T} . This will imply that the space of Drinfeld modular forms tensored with \mathbb{T} is equal to the space of VDMFs corresponding to $\mathbf{1}$. Thus one needs to focus on the higher dimension case to produce non-trivial examples. For this aim, we define the functions \mathcal{F} and \mathcal{F}^* given in [Pel12, §2.2, 2.3], which are examples of weak VDMFs of dimension two constructed by using Anderson generating functions. We also define the deformation of the Eisenstein series. We

will finalize the talk with a sketch of the proof of [Pel12, Prop. 19]. The main references are [Pel12, §1,2] and [Pel14, §2,3].

Talk 4: Special values of L -functions

Speaker: Alireza Shavali Kohshor

Date: May 16th, 2023

In this talk, we focus on how to determine special values of twisted Goss L -functions as given in [Pel12, Thm. 1, Thm. 8] by using weak vectorial Drinfeld modular forms. For the convenience of the audience, the talk should start with some brief introduction to Goss L -functions [Gos96, §8] (see also [CEGP18, §3]). Later on, we focus on [Pel12, Prop. 26] which reveals the behavior of certain \mathbb{T} -valued functions at the cusp at infinity. Then using previous results proved in Talk 3 as well as [Pel12, Prop. 26], a proof for [Pel12, Thm. 8] must be given. Moreover the statement of [Pel12, Thm. 1] as well as its consequences should be discussed. If time permits, a proof of [Pel12, Thm. 1] will be given.

Talk 5: Vectorial Drinfeld modular forms

Speaker: Gebhard Böckle

Date: May 23th, 2023

Our fifth talk is to start investigating VDMFs as well as discussing their several properties which will be later used to reveal some applications for Drinfeld modular forms. They are weak VDMFs corresponding to a certain character with a regularity condition introduced in [PP18, Def. 3.4(2)]. After defining VDMFs, we revisit the deformation of Eisenstein series and prove [PP18, Prop. 3.7] which gives the Fourier expansion of their each entry (see also [Pel12, Lem. 21]). An equivalent condition for the regularity [PP18, Cor. 2.6] should also be analyzed. Later on, we introduce the function \mathcal{F} discussed in the previous talk and show that it is indeed not a VDMF in the sense of [PP18]. The final goal is to prove [PP18, Thm. 3.9] which allows one to decompose a certain space of VDMFs into components generated by an Eisenstein series \mathcal{E}_1 and its twist, the image of \mathcal{E}_1 under the q -th power Frobenius automorphism of \mathbb{T} .

Talk 6: Drinfeld modular forms of prime power levels via vectorial modular forms

Speaker: Sriram Chinthlagiri Venkata

Date: May 30th, 2023

We now focus on another application of VDMFs for obtaining Drinfeld modular forms of certain level. This will be done by evaluating \mathbb{T} -valued functions on a particular point of \mathbb{C}_∞ as well as taking the hyperderivatives of entries of VDMFs. More precisely, the talk should cover the content of [PP18, Prop. 4.11] and [PP18, Prop. 4.19]. Hence it should be organized so that all necessary background, such as Drinfeld modular forms for congruence subgroups and notion of hyperderivatives, to achieve the above mentioned results are introduced. The main references are [PP18, §4.1–4.2.4] and [PP18, §4.4–4.4.2].

Talk 7: Hecke operators

Speaker: Oğuz Gezmiş

Date: June 6th, 2023

Our goal in the last talk is to study Hecke operators acting on the space of VDMFs and consequences of such an action for Drinfeld modular forms. In particular, applying hyperderivatives and evaluating the coordinates of vectorial modular forms at roots of unity, we will obtain Hecke eigenforms for certain congruence subgroups of $GL_2(A)$. We start with defining the slash operator and the Hecke operator T_p for each monic irreducible polynomial

\mathfrak{p} in A . Later on, we give the necessary ideas for the proof of [PP18, Prop. 5.12, Prop. 5.18] and briefly mention why the regularity condition is required for the stability of the space of VDMFs under Hecke operators [PP18, Rem. 5.13]. We continue with giving some examples of Hecke eigenforms in our setting and analyze the behavior of vectorial Eisenstein series under $T_{\mathfrak{p}}$ as well as provide a sketch of a proof for [PP18, Cor. 5.23-5.24]. The main reference for the talk is [PP18, §5–5.3].

REFERENCES

- [BP20] W. D. Brownawell and M. A. Papanikolas, *A rapid introduction to Drinfeld modules, t -modules, and t -motives*, *t -Motives: Hodge Structures, Transcendence, and Other Motivic Aspects*, European Mathematical Society, Zürich, 2020, pp. 3-30.
- [CEGP18] C.-Y. Chang, A. El-Guindy, and M. A. Papanikolas, *Log-algebraic identities on Drinfeld modules and special L -values*, *J. London Math. Soc. (2)* **97** (2018), 125–144.
- [EGP14] A. El-Guindy and M. A. Papanikolas, *Identities for Anderson generating functions for Drinfeld modules*, *Monatsh. Math.* **173** (2014), no. 3–4, 471–493.
- [FvdP04] J. Fresnel and M. van der Put, *Rigid Analytic Geometry and its Applications*, Birkhäuser, Boston (2004).
- [Gek88] E.-U. Gekeler, *On the coefficients of Drinfeld modular forms*, *Invent. Math.* **93**, No: 3, (1988), 667–700.
- [Gos80a] D. Goss, *π -adic Eisenstein series for function fields*, *Compos. Math.* **41** (1980), no. 1, 3–38.
- [Gos80b] D. Goss, *The algebraists upper half-plane*, *Bull. Amer. Math. Soc. (N.S.)* **2** (1980), no. 3, 391–415.
- [Gos92] D. Goss, *A short introduction to rigid analytic spaces*, in: *The Arithmetic of Function Fields (Proceedings of the Workshop at the Ohio State University, June 1726, 1991)*, pp. 265–284, edited by D. Goss, D. R. Hayes and M. I. Rosen, Walter de Gruyter, Berlin, 1992.
- [Gos96] D. Goss, *Basic Structures of Function Field Arithmetic*, Springer-Verlag, Berlin, 1996.
- [Pel08] F. Pellarin, *Aspects de l'indépendance algébrique en caractéristique non nulle*, *Sém. Bourbaki*, vol. 2006/2007. Astérisque **317** (2008), no. 973, viii, 205–242.
- [Pel12] F. Pellarin, *Values of certain L -series in positive characteristic*, *Ann. of Math. (2)* **176** (2012), no. 3, 2055–2093.
- [Pel14] F. Pellarin, *Estimating the order of vanishing at infinity of Drinfeld quasi-modular forms*, *J. Reine Angew. Math.* **687** (2014), 1-42.
- [PP18] F. Pellarin and R. B. Perkins, *Vectorial Drinfeld modular forms over Tate algebras*, *Int. J. Number Theory* **14**, 2018, 1729–1783.
- [Pel21] F. Pellarin, *The analytic theory of vectorial Drinfeld modular forms*, arXiv:1910.12743, 2021.
- [Rev92] M. Reversat, *Lecture on rigid geometry*, in: *The Arithmetic of Function Fields (Proceedings of the Workshop at the Ohio State University, June 1726, 1991)*, pp. 265–284, edited by D. Goss, D. R. Hayes and M. I. Rosen, Walter de Gruyter, Berlin, 1992.
- [Ste97] G. Van Steen, *Some rigid geometry*. In *Drinfeld modules, modular schemes and applications (Alden-Biesen, 1996)*, World Sci. Publ., River Edge, NJ, 1997, pp. 88–102.