MODULI OF CURVES AND THE THEORY OF ALGEBRAIC STACKS

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1. Introduction

This seminar is aimed at advanced master's students and PhD students eager to learn a bit more about algebraic geometry through the use of moduli spaces.

The goal of this seminar is to familiarize its participants with a modern (hence stacky) perspective on the theory of moduli spaces. Note that moduli spaces are ubiquitous in mathematics. In fact, one of the very first varieties we encounter in life, namely $\mathbb{P}^1(\mathbb{C})$, is a moduli space (parametrizing linear subspaces of $\mathbb{C}^2$).

We will try to understand why moduli spaces usually do not exist (with all of its desired properties) as algebraic varieties, but do exist as algebraic stacks. We will then explain how certain properties of a moduli space (e.g., being separated, proper, smooth, connected, or "Deligne-Mumford") translate back into properties of the objects that the moduli space parametrizes.

We will see that many natural moduli problems can be represented (in a meaningful way) by algebraic stacks, e.g., moduli of smooth curves, polarized abelian varieties, hypersurfaces in projective space., and the moduli of Fano varieties.

In the previous semester the seminar discussed descent theory and ended with introducing the notion of a stack (i.e., groupoid-valued sheaf on the category of schemes). **We will not assume the participants of this seminar were present at the previous semester.**

2. Program

Each of the following six topics constitutes essentially two talks of about 90 minutes approximately.

2.1. The moduli space of smooth curves.
Aim of this talk is to get a feeling for the diagonal of a moduli stack. The diagonal of a moduli stack "sees" in a precise sense the properties of automorphism groups of the objects it parametrizes.

Define smooth proper curves over arbitrary schemes. Explain that the moduli functor M_g is not representable. Recall briefly what a stack is and that $\mathcal{M}_g$ is a stack. (Do not prove that $\mathcal{M}_g$ is a stack to avoid too much repetition. In any case, we will prove this in the next talks in a more general setting.) Explain the relation between the diagonal of a stack and isomorphisms between two objects of the stack.

Define what it means for a morphism of stacks to be representable, schematic, proper, and unramified, respectively.

Assume $g > 1$. Prove that the diagonal of $\overline{\mathcal{M}}_g$ (or just $\mathcal{M}_g$) is schematic, proper, and unramified. Insist that this translates into the statement that automorphism groups of curves are finite reduced group schemes, and that the Isom-scheme between two smooth proper curves of genus g ($g > 1$) is proper.

If time permits, show that a smooth proper curve over some base curve B is *isotrivial* (i.e., all fibres are isomorphic) if and only if it is trivial after a finite étale base change of B.

- [1]
- [2, Chapter 13]

2.2. **The moduli space of polarized varieties.** There is not an "algebraic" object parametrizing ***all*** varieties, not even all smooth projective ones. One way out of this is to study varieties together with an ample line bundle. The existence of an "algebraic" moduli space parametrizing all such objects then follows from the theory of Hilbert schemes. Rougly speaking, the additional datum of an ample line bundle can be used to embed your variety into some projective space (from which the connection to Hilbert schemes becomes clear).

Introduce notion of f-ample bundle. Define polarized (smooth proper) varieties and families of polarized varieties. Recall that the moduli of polarized varieties is a stack, and give an idea of the proof (following Olsson). Prove that the diagonal of this stack is schematic. Insist that this follows from the statement about isom-schemes between polarized varieties are affine algebraic groups.

Show that the diagonal of this stack is not necessarily proper. (Look at $\mathcal{M}_0$.) (Thus, with terminology we introduce later, this means that its connected components are not necessarily separated.)

Focus on the stack of canonically polarized varieties (those with ample canonical bundle). Explain that the diagonal here is proper (and even finite). Show that it is not necessarily unramified in characteristic $p > 0$ (which shows that the analogous statement about the diagonal of $\mathcal{M}_g$ when $g > 1$ is quite specific to curves). Explain that is always unramified in characteristic zero, and how this relates to Cartier's theorem that group schemes over fields of characteristic zero are reduced (hence smooth).

Introduce the moduli stack of principally polarized abelian varieties $\mathcal{A}_g$ and show that it has a finite unramified diagonal (over $\mathbb{Z}$). Here the fact that this is a stack follows from the more general fact that the "stack of polarized varieties" is a stack. The claimed properties of the diagonal of $\mathcal{A}_g$ translate into the following statement.

Theorem. Let S be a scheme and let $A \to S$ and $B \to S$ be principally polarized abelian schemes. Then the scheme of isomorphisms $Isom_S(A, B)$ parametrizing isomorphisms $A \to B$ of principally polarized abelian schemes over S is finite unramified over S.

Here isomorphisms are understood to respect the group structure on A and B, as well as the polarizations (surpressed in the notation).

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2.3. **Rigidifying moduli problems.** The moduli functor of elliptic curves (resp. smooth proper curves of genus g) is not representable by a scheme. By changing a moduli problem slightly, one sometimes can prove the existence of a *fine* moduli space which, in some suitable sense, is quite close to the moduli functor.

The simplest example is arguably given by the modular curves $Y(n)$ (with n a positiv integer). For each positive integer n, the functor $Y(n)$ associates to a scheme S over $\mathbb{Z}[1/n]$ the set of isomorphism classes of elliptic curves over S endowed with a full level N-structure (i.e., the choice of a basis for the n-torsion). For $n > 2$, the functor is representable by a smooth affine curve over $\mathbb{Z}[1/n]$. The functor $Y(1)$ is not representable, but is (famously) coarsely represented by the j-line $\mathbb{A}^1$; the notion of being coarsely represented will be discussed in the Bonus Talk if time permits. For each $n > 2$, the stack $\mathcal{Y}(1)$ of elliptic curves comes equipped with a morphism $Y(n) \to \mathcal{Y}(1)$, and this morphism is (representable) finite étale (in the category of stacks).

Another good example of "rigididying a moduli space" is given by the moduli of principally polarized ableian varieties (ppav's) with level N structure ($N > 3$); this is a higher-dimensional version of what we discussed before for elliptic curves. The stack $\mathcal{A}_g$ is a quotient of the scheme $\mathcal{A}_g^n$ by a finite group. It is a good example of a quotient stack $[X/G]$, with G a finite group. Here the "rigidified moduli problem" is again finite étale over the original moduli problem (of principally polarized abelian varieties).

Start this talk by explaining the above and proving the claimed statements. If time permits, also discuss the Hilbert scheme of smooth curves in $\mathbb{P}^N$. Explain that $\mathcal{M}_g$ is a quotient of this Hilbert scheme by some (infinite) algebraic group. Explain that the Hilbert scheme is a "rigidified" version of the moduli functor of curves.

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2.4. **Algebraic spaces and algebraic stacks.** In this talk we (finally) introduce the notion of an algebraic stack. To do this, we need the notion of an algebraic space. Despite its "name", an algebraic space is (only) a sheaf.

Introduce fppf sheaves, and define algebraic spaces after stating that (the functor of points of) a scheme is an fppf sheaf. The latter is something we have already seen by now, so we won't need a proof.

Explain that schemes are algebraic spaces (since schemes are fppf sheaves and the identity map is an etale atlas). Explain that there are algebraic spaces which are not schemes (in dimension > 1) citing Hironaka's famous example. Also explain that there are algebraic spaces which are not of the form X/G (with G acting freely and X is a scheme). (These are not trivial to construct. Do not dwell too long on the details in these constructions.)

Define representable morphisms, and what it means for a representable morphism of stacks to be smooth and surjective. Define algebraic stacks following the stacks project (asking only for representability of the diagonal and the existence of a smooth surjection from a scheme). Define what a morphism of algebraic stacks, and repeat what it means to be representable.

Emphasize that algebraic spaces (and schemes) are algebraic stacks, and that they are not necessarily quotients of schemes by finite groups.

Explain why the stack $\mathcal{M}_g$ has a smooth presentation by a scheme (using Hilbert schemes), same for $\overline{\mathcal{M}_g}$. Same for $\mathcal{A}_g$ if time permits. Conclude that $\mathcal{M}_g$ and $\mathcal{A}_g$ are algebraic stacks.

Define a stack with representable diagonal to be Deligne-Mumford if it has an étale presentation. Note that such stacks are (obviously) algebraic. Show that an algebraic stack is Deligne-Mumford if and only if its diagonal is formally unramified (and explain what the latter means for automorphism groups of moduli stacks). Apply this to $\mathcal{M}_g$ and the other stacks we have seen so far, and conclude that $\mathcal{M}_g$ is a Deligne-Mumford stack.

Define separatedness of stacks via the diagonal and explain that the results in the first talks can be reformulated as saying that the respective moduli stacks are separated.

- [2] chapter 13.1)
- [2] chapter 5.1-5.3 + 8.3
- [7],
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2.5. **Algebraic stacks and Artin's axioms.** Discuss Artin's axioms for schemes.

Explain that a stack is algebraic if and only if it satisfies Artin's axioms. Use this to show that the stack of abelian varieties is not algebraic. (This is why we need polarizations!) Same argument can be used for K3 surfaces. (Emphasize that each of Artin's axioms translates back into some property of the objects that the stack parametrizes.)

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2.6. **Smoothness of stacks vs unobstructedness of objects.** Let M be a moduli stack (e.g., the moduli of smooth curves of genus g, stable curves of genus g, polarized abelian varieties, smooth hypersurfaces, or "very singular" curves, etc.)

Explain that smoothness of the stack translates (via the infinitesimal lifting criterion) into a statement about liftability of objects. Define tangent spaces of stacks and compute them using cohomology groups.

Show that $\mathcal{M}_g$ is smooth by "counting" its dimension and by computing $h^1(X, T_X)$ for X a smooth projective curve of genus g.

Same for moduli stack of smooth hypersurfaces and polarized abelian varieties. Talk about unobstructedness of Calabi-Yau varieties (or K3 surfaces) and polarized Calabi-Yau varieties and how this translates into smoothness of the respective moduli stacks.

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2.7. **Coarse spaces.** Bonus talk. Certain non-representable functors (such as M_g) are "coarsely representable". Explain this for M_g. (The construction of M_g can be done using GIT to obtain that it is a quasi-projective scheme.) Then state the general theorem of Keel-Mori that gives the existence of the coarse space M_g as an algebraic space simply from the fact that $\mathcal{M}_g$ is a finite type separated algebraic stack with finite diagonal.

References will be added in due course.

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