OBERSEMINAR:
THE WITT VECTOR AFFINE GRASSMANNIAN

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The goal of this seminar is to understand Bhatt and Scholze's paper [BS17]. The basic question that is discussed in this paper is the following: Let M be the set of $\mathbb{Z}_p$-lattices in $\mathbb{Q}_p^n$ or, equivalently, the quotient set $\mathrm{GL}_n(\mathbb{Q}_p)/\mathrm{GL}_n(\mathbb{Z}_p)$. Is there a canonical algebro-geometric structure on M ? This question is answered affirmatively in [BS17]. More precisely, they show that the functor $\mathrm{Gr}^{\mathrm{Waff}}$ on perfect $\mathbb{F}_p$-algebras that sends R to $\mathrm{GL}_n(W(R)[\frac{1}{p}])/\mathrm{GL}_n(W(R))$ is representable by an inductive limit of perfections of projective $\mathbb{F}_p$-schemes. Here $W(R)$ denotes the ring of Witt vectors of the perfect $\mathbb{F}_p$-algebra R (recalled in Talk 6).

Such sets of lattices appear for example in the Langlands program. The result of the paper is interesting as it allows us to define important invariants, like étale cohomology, attached to these sets.

The geometry of perfect schemes in characteristic p and its surprising properties play an essential role in the proof of the main result of [BS17]. These will be explored in Talks 5, 7–10. For instance we will prove very strong descent results for vector bundles on perfect schemes, which fail in the usual, non-perfect world. The necessary background about descent and Grothendieck topologies in general will be provided in the earlier talks.

Dates and location: Tuesday, 14-16

Talks: The talks marked with an asterisk essentially don't depend on the other talks and should be accessible for PhD or MSc students.

(1)* *Grothendieck topologies and flat descent* — Introduce the notion of a Grothendieck topology using coverings (or covering sieves, if you prefer) and sheaves. Example: fppf topology. Discuss faithfully flat descent of quasi-coherent modules. This can be interpreted as a sheaf property for the fppf topology. References: [Del77, Arcata, I], [BLR90, §6.1], [Bos13, §4.6]

(2) *The h-topology* — Introduce v-covers and the h-topology as in [BS17, §2]. Discuss Examples of v-covers (in particular, faithfully flat morphisms and proper surjective morphisms). Prove the sheaf-criterion Prop. 2.8.

(3) *Interlude: ∞-categories and sheaves of spaces* — Give a brief introduction to ∞-categories. In particular, we should get a feeling for spaces, limits, (pre)sheaves on a classical Grothendieck site with values in spaces. One could also discuss spectra and explain how sheaf cohomology is just sheafification in the ∞-categorical sense.

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(4) *h-sheaves of spaces* — Extend the sheaf-criterion from Talk 2 to sheaves of spaces and discuss applications: [BS17, 2.9–2.14].

(5)* *Perfect schemes* — Discuss various results about perfect schemes in characteristic p and the perfection functor as in [BS17, §3], in particular: Lemmas 3.4, 3.8, and Prop. 3.13. Lemmas 3.16 and 3.18 may also be deferred to Talk 9.

(6)* *Interlude: Witt vectors* — Introduce Witt vectors of perfect rings of characteristic p as in [Ser79, §II.6] or Bergman's lecture [Mum66, Lecture 26].

(7–8) *v-descent on perfect schemes* — Prove that the v-topology on perfect schemes is subcanonical and that vector bundles on affines have no higher cohomology. Prove v-descent for vector bundles on perfect schemes. References: [BST17, §3], [BS17, §4].

(9–10) *Fiber-wise criteria* — Discuss [BS17, §6, 6.1–6.11]. Here we use the perfect base change formula from Lemmas 3.16 and 3.18. Maybe also recall some necessary facts about valuation rings. The important results are 6.1 and 6.8–6.11.

(11)* *Families of torsion $W(k)$-modules* — Discuss basics about p-power torsion $W(R)$-modules for perfect rings R : [BS17, 7.1–7.8].

(12) *The Demazure scheme* — Prove the existence of the Demazure scheme. For this provide some necesssary facts about Grassmannians and the Quot-scheme. [BS17, 7.9–7.14]

(13) *The Witt vector affine Grassmannian* — Introduce the functors $\mathrm{Gr}_{\leq\lambda}$ and the Demazure resolution $\widetilde{\mathrm{Gr}}_\lambda$. Prove that the Demazure resolution is representable and that the canonical line bundle on $\widetilde{\mathrm{Gr}}_\lambda$ descends to $\mathrm{Gr}_{\leq\lambda}$. [BS17, 8.1–8.8].

(14) *Projectivity of the Witt vector affine Grassmannian* — Prove that each $\mathrm{Gr}_{\leq\lambda}$ is representable by the perfection of a projective $\mathbb{F}_p$-scheme: [BS17, §8.4].

References

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[Mum66] David Mumford, *Lectures on curves on an algebraic surface*, Annals of Mathematics Studies, No. 59, Princeton University Press, Princeton, N.J., 1966, With a section by G. M. Bergman. MR 0209285

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