SEMINAR ON ÉTALE MOTIVES, SUMMER 2022

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This is a learning seminar on Cisinski–Déglise's paper [CD16]. We study the construction of triangulated categories of étale motives, their formal properties encoded in the six functors formalism and their relation with derived categories of torsion étale sheaves, with Chow groups and K-theory.

The participants are referred to [CD19, Introduction A] for an overview on the development of the subject and to [Cis21] for a recent survey. Text book accounts include the books of André [And04] for the general theory of motives and Mazza–Voevodsky–Weibel [MVW06] for motivic cohomology.

General information. Each talk is 90 minutes long, starting Tuesdays at 14:00 MEST (sharp) in Room S215 401 at TU Darmstadt. The seminar is held in hybrid format. The talks will be streamed online but will not be recorded. All participants are encouraged to ask many questions, before, during and after the seminar.

Talk 0: Leitfaden, April 12. This is a short talk (about 15 minutes) given by Timo and intended as an overview of the program.

1. Prelude on étale cohomology

Étale cohomology, especially with torsion coefficients, serves as the guiding example throughout the seminar.

Talk 1: Review of étale cohomology, April 19. Follow [DB77, I–III]: for a scheme X and a (commutative, unital) ring R, introduce the category $\mathrm{Sh}(X_{\text{ét}}, R)$ of sheaves of R-modules on the small étale site $X_{\text{ét}}$, the basic functorialities $f_\sharp, f^*, f_*$ for $f\colon Y \to X$, define cohomology and explain what it computes in basic examples (Galois cohomology, coefficients in $\mathbb{G}_m$, $\mathbb{G}_a$). Use this to compute $H^*(X_{\text{ét}}, \mathbb{F}_p)$ for affine schemes X of characteristic $p > 0$ (Artin-Schreier sequence), $\mathrm{H}^*_{\text{ét}}(X, \mathbb{Q})$ if X is normal, Noetherian and connected [nfc17] and sketch the computation of the cohomology of curves using the Kummer sequence [DB77, Arcata, Corollaire 3.5]. Specialize to the case $R = \mathbb{Z}/n$ with $n \in \mathcal{O}_X^\times$, define Tate twists and deduce $\mathrm{H}^*_{\text{ét}}(\mathbb{A}^1_k, \mathbb{Z}/n) = \mathbb{Z}/n[0]$ and $\mathrm{H}^*_{\text{ét}}(\mathbb{P}^1_k, \mathbb{Z}/n) = \mathbb{Z}/n[0] \oplus \mathbb{Z}/n(-1)[-2]$ for an algebraically closed field k with $n \in k^\times$ (compare this with the cases $R = \mathbb{F}_p, \mathbb{Q}$). Time permitting present *elements* of the six functors formalism for the derived category $\mathrm{D}(X_{\text{ét}}, \mathbb{Z}/n)$: for morphisms $f\colon Y \to X$, pairs of adjoint functors (Rf_*, f^*) and, if f is separated of finite type $(Rf^!, Rf_!)$, and $(\underline{R\mathrm{Hom}}(\text{-},\text{-}), \text{-} \otimes^{\mathbb{L}} \text{-})$ satisfying favorable properties, see [CD16, Definition A.1.10] for a summary and [BS15, Section 6.7] for proofs.

2. Axiomatic approach to the six functors formalism

This follows the summary in [CD16, Appendix A.1]. Details are in [CD19, Sections 1, 2, 4]. See also [Gal21] for a general introduction to six functors formalisms.

Talk 2: Premotivic categories, April 26. Recall fibered categories [CD19, Section 1.1]. Introduce abelian and triangulated premotivic categories [CD16, Definition A.1.1] (see also [CD19, Section 1.4] for more details and [CD19, Sections 1.1.9, 1.1.26] for the definition of the exchange transformations). Discuss the examples in [CD16, Examples A.1.3] and explain how $p_\sharp$ is constructed. Follow [CD16, A.1.4]: define the premotive of a scheme, explain the formulas [CD19, Section 1.1.37], introduce Tate twists (see also [CD19, Definition 2.4.17]) and the associated bigraded cohomology. State [CD16, Definition A.1.5] and explain why these properties hold in the case of $\mathscr{T} = \mathrm{D}(X_{\text{ét}}, \mathbb{Z}/n) = \mathrm{D}(\mathrm{Sh}_{\text{ét}}(\mathrm{Et}_X, \mathbb{Z}/n))$ with $n \in \mathcal{O}_X^\times$. Cover the material in [CD16, A.1.6–A.1.8].

Talk 3: The six functors, May 3. Give the definition of a (oriented) six functors formalism [CD16, Definition A.1.10] (the notion of orientation will be discussed further in Talk 4). Introduce the (weak) localization and purity properties [CD16, Section A.1.11, Definition A.1.12]. State [CD19, Theorem A.1.13] (such categories are called motivic [CD16, Definition A.1.15], [CD19, Definition 2.4.45, Remark 2.4.27]) and sketch its proof following [CD19, Theorem 2.4.50]: construct the exceptional functors [CD19, Propositon 2.2.7] and prove [CD19, Proposition 2.2.10] (time permitting [CD19, Corollary 2.2.12, Parts 1., 3.]); discuss the localization property [CD19, Proposition 2.3.3, Corollary 2.3.13]; define the Thom transformation [CD19, Definition 2.4.1], sketch [CD19, Corollaries 2.4.14, 2.4.19], construct $\mathfrak{p}_f$ [CD19, Sections 2.4.20] and prove [CD19, Theorem 2.4.26]. Mention the form [CD19, (2.4.37.1)] of $\mathfrak{p}_f$, the notion of orientation [CD19, Definition 2.4.38] in order to get the usual Tate twists [CD19, (2.4.39.3)]. Finish by stating [CD19, Theorem 2.4.28] and comment on its proof (time permitting).

Talk 4: Constructible motives, May 10. Start with a review of perfect-constructible objects in $\mathrm{D}(X_{\text{ét}}, R)$ [BS15, Sections 6.3, 6.4]. Recall that under cohomological finiteness assumptions on $(X_{\text{ét}}, R)$, an object is perfect-constructible if and only if it is compact [BS15, Proposition 6.4.8], compare with [CD19, Proposition 1.4.11]. Follow [CD19, Section 4.2]: define constructible (pre-)motives, give [CD19, 4.2.2–4.2.5] and also [CD19, 4.2.6–4.2.13] as time permits; state [CD19, Theorem 4.2.29] and sketch its proof.

3. Construction of premotivic categories

This section follows [CD19, Sections 5.1–5.3]. The references [CD16, CD19] are written in the language of model categories. The speakers are invited to make the link with ∞-categories, see [Cis21, Section 1.2] and [Rob15, Sections 2.1, 2.2].

Talk 5: Interlude on model categories, May 17. A reference is [DS95]. Define model categories. Explain the projective model category structures on complexes of R-modules [DS95, Example 3.7] and (time permitting) its generalization to complexes of sheaves of R-modules [CD16, Example 1.1.8]. Relate fibrations and cofibrations with liftings of the other. Define the homotopy category of a model category and show that it is the same as the localization with respect to weak equivalences. Make it explicit by looking at the model category of chain complexes. Define derived functors and relate them to the classical theory of derived functors in the derived category. Define Quillen adjunctions/equivalences and (time permitting) the relation to ∞-categories [Lur09, Section A.2 (2)].

Talk 6: $\mathbb{A}^1$-localization, May 24. This follows [CD19, Sections 5.1, 5.2]. Explain how to associate to a Grothendieck abelian premotivic category $\mathscr{A}$ the model category $C(\mathscr{A})$ of complexes over it [CD19, Proposition 5.1.12] and the triangulated premotivic category $\mathrm{D}(\mathscr{A})$. Discuss the example of presheaves and sheaves in an admissible topology. Mention that the construction is functorial in $\mathscr{A}$ and discuss [CD19, Example 5.1.24]. Given a small family of morphisms $\mathscr{W}$ in $C(\mathscr{A})$, generalize the previous to the $\mathscr{W}$-local model category structure and show that $\mathrm{D}(\mathscr{A})[\mathscr{W}^{-1}]$ identifies with the Verdier quotient $\mathrm{D}(\mathscr{A})/\langle\mathscr{W}\rangle$, see [Lur09, A.3.7.4, A.3.7.6] for the relation between Bousfield localization in model categories and ∞-categories. Deduce the adjunction $\mathrm{D}(\mathscr{A}) \leftrightarrows \mathrm{D}^{\mathrm{eff}}_{\mathbb{A}^1}(\mathscr{A})$ of triangulated premotivic categories and define strong $\mathbb{A}^1$-equivalence and $\mathbb{A}^1$-contractibility of complexes. Time permitting sketch the construction of explicit $\mathbb{A}^1$-resolutions [CD19, 5.2.c]. End with the example $\mathrm{D}^{\mathrm{eff}}_{\mathbb{A}^1}(\mathrm{Sh}_{\text{ét}}(\mathrm{Sm}_X, R))$ and show [CD16, Proposition A.3.1].

Talk 7: $\mathbb{P}^1$-stabilization, May 31. This follows [CD19, Section 5.3]. Introduce spectra and construct the premotivic adjunction $\Sigma^\infty : \mathrm{D}^{\mathrm{eff}}_{\mathbb{A}^1}(\mathscr{A}) \leftrightarrows \mathrm{D}_{\mathbb{A}^1}(\mathscr{A}) : \Omega^\infty$, see [Rob15, Section 2.2] for the relation with ∞-categories. Deduce that $\mathrm{D}_{\mathbb{A}^1}(\mathscr{A})$ satisfies the homotopy and stability properties (and descent). Mention the universal example [CD19, Remark 5.3.34]. Make this concrete for $\mathrm{D}_{\mathbb{A}^1}(\mathrm{Sh}_{\text{ét}}(\mathrm{Sm}_X, R))$ in [CD19, Example 5.3.31] and mention it is orientable [CD16, Corollary 5.5.7, Section 5.6.1].

4. Categories of étale motives

This follows [CD16, Sections 2–5].

Talk 8: Interlude on finite correspondences, June 7. Introduce cycles, cycles associated to a closed subscheme, pushforward and flat pullback [Ful98, Sections 1.4, 1.5, 1.7]. Introduce the Serre Tor-formula for the intersection of cycles [Ful98, Section 20.4], [Sta21, 0AZR] (see also [MVW06, Appendix 17.A]) and state Serre's result/conjecture [Ser00, Section V, B.3–4] (conjecture is known today). Define the category of finite correspondences $\mathrm{Sm}_k^{\mathrm{cor}}$, the graph functor $\mathrm{Sm}_k \to \mathrm{Sm}_k^{\mathrm{cor}}$ and prove properties as time permits [MVW06, Lecture 1].

Talk 9: Étale sheaves with transfers, June 14. This follows [CD16, Section 2]. Briefly state the extension to the relative setting of the category of finite correspondences [CD19, Section 9]. Mention that the composition of finite correspondences is still given by Serre's Tor formula over regular bases [CD19, Remark 9.1.6]. Introduce étale sheaves with transfers $\mathrm{Sh}^{\mathrm{tr}}_{\text{ét}}(\text{-}, R)$ which forms a Grothendieck abelian premotivic category [CD16, Proposition 2.1.10, Corollary 2.1.12]. Show it is compatible with the étale topology (so Talks 6&7 apply) and define $\mathrm{DM}^{\mathrm{eff}}_{\text{ét}}$ and $\mathrm{DM}_{\text{ét}}$, see [CD16, Section 2.2.4].

Talk 10: Torsion étale motives, June 21. This follows [CD16, Sections 3, 4]. Let R be a torsion ring. Explain that $\mathrm{DM}^{\mathrm{eff}}_{\text{ét}}(\text{-}, R)$ satisfies the stability property [CD16, Corollary 4.1.2]. Show that $\mathrm{DM}_{\text{ét}}(\text{-}, R)$ is orientable and satisfies the localization property [CD16, Section 4.1.3, Theorem 4.3.1]. Deduce that it satisfies the Grothendieck six functors formalism. Explain the proof of the rigidity theorem [CD16, Theorem 4.5.2, Corollaries 4.5.3, 4.5.4].

Talk 11: Motives and h-descent, June 28. This follows [CD16, Section 5]. Recall the h-topology and introduce the categories of h-motives [CD16, Section 5.1]. Prove the comparison with torsion étale motives [CD16, Theorem 5.5.3, Corollary 5.5.4] from the previous talk. End by stating the comparisons [CD16, Corollaries 5.5.5, 5.5.7] and the six functors formalism for h-motives [CD19, Theorem 5.6.2] (the proof for $\mathbb{Q}$-algebras relies on the comparison with Beilinson motives addressed in Talks 14&15).

5. Comparison with rationalized Chow groups

This follows [MVW06] where the base scheme is a (perfect) field. Be aware that the language sometimes differs from [CD19].

Talk 12: Interlude on Chow groups, July 5. This follows [MVW06, Lecture 17]. Define higher Chow groups and present its properties. Show, as time permits, that higher Chow groups have the structure of presheaves with transfers.

Talk 13: Comparison with rationalized Chow groups, July 12. Outline the proof that the Hom-groups in $\mathrm{DM}_{\text{ét}}(X, \mathbb{Q})$ compute rationalized higher Chow groups. For this, recall the Nisnevich topology and define motivic cohomology as Hom-groups in $\mathrm{DM}^{\mathrm{eff}}(X, R) := \mathrm{D}^{\mathrm{eff}}_{\mathbb{A}^1}(\mathrm{Sh}_{\mathrm{Nis}}(\mathrm{Sm}_X, R))$, see [CD19, Section 11.2.a]. Show that the comparison map $\mathrm{DM}^{\mathrm{eff}}(X, R) \to \mathrm{DM}^{\mathrm{eff}}_{\text{ét}}(X, R)$ is an equivalence for any $\mathbb{Q}$-algebra R, see [CD16, Proposition 2.2.10]. So the Hom-groups in $\mathrm{DM}_{\text{ét}}(X, \mathbb{Q})$ compute rationalized motivic cohomology. For a smooth separated scheme over a perfect field, show that the so defined motivic cohomology groups agree with Voevodsky's definition [CD19, Example 11.2.3] and outline the proof of the comparison theorem [MVW06, Theorem 19.1].

6. Comparison with algebraic K-theory

The final session will be a double session on the comparison with (rationalized) algebraic K-theory. We hope to welcome all participants of the seminar in person in Darmstadt.

Talk 14: Interlude on algebraic K-theory, July 19, 14:00-15:30. Explain the approach to algebraic K-theory using spectra. Introduce the K-theory spectrum[1] and modules over it [CD19, Section 13]. Introduce the γ-filtration [CD19, Section 14.1] and use it to define the Beilinson motivic cohomology spectrum and Beilinson motives. Say a few words about the relation between Beilinson motivic cohomology and algebraic K-theory [CD19, Corollary 14.2.14].

Coffee break.

Talk 15: Comparison with algebraic K-theory, July 19, 16:15-17:45. Say as much as possible about the proof of the comparison between h-motives and Beilinson motives [CD16, Theorem 5.2.2]. Discuss the localization property for Beilinson motives, and sketch how combining previous results imply the full six functors formalism for h-motives with arbitrary coefficients, compare with Talk 11.

Abschlussfest.

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[1] It is suggested to replace all instances of SH by $\mathrm{D}_{\mathbb{A}^1,\mathbb{Q}}$, see [CD19, Section 12.3].