THE ANDRÉ-OORT CONJECTURE

GAUS AG-Seminar – SoSe 2022

ORGANIZERS: JIAMING CHEN, JAKOB STIX, RICCARDO ZUFFETTI

Shimura varieties are algebraic varieties of great interest. Introduced by Shimura and Deligne in order to generalize the modular curves, they play nowadays a central role in the theory of automorphic forms, the study of Galois representations and in Diophantine geometry. The classical example of Shimura variety is the Siegel moduli space $\mathcal{A}_g$ of principally polarized abelian varieties of dimension g. The points in this moduli space corresponding to CM abelian varieties, which are called *special points*, play a particularly important role in the theory of $\mathcal{A}_g$. The concept of special points have been generalized to arbitrary Shimura varieties and every Shimura variety has a Zariski (even analytically) dense subset of special points which, as for $\mathcal{A}_g$, governs its geometry and arithmetic. This is justified for example in the theory of canonical models over number fields of Shimura varieties.

The André-Oort conjecture describes the distribution of special points on a Shimura variety:

Conjecture 0.1 (André-Oort conjecture). *Let V be a closed irreducible subvariety of a Shimura variety S. If V contains a Zariski-dense subset of special points of S, then V is special, i.e., V is an irreducible component of a Hecke translate of a Shimura subvariety of S.*

The André-Oort conjecture is a Hodge-theoretic analog of the Manin-Mumford conjecture. The latter, which has been proved by Raynaud, states that an irreducible subvariety of a complex abelian variety containing a Zariski-dense subset of torsion points is a torsion translate of an abelian subvariety.

Conjecture 0.1 was resolved recently in full generality by Pila-Shankar-Tsimerman [PST21] following a strategy proposed by Pila and Zannier, which has three main ingredients:

(i) definability in some o-minimal structure of some transcendental map;
(ii) functional transcendence for Shimura varieties;
(iii) good lower bound for the size of Galois orbits of special points.

Once having these ingredients, assembled by the o-minimality through the Pila-Wilkie counting theorem, one can conclude Conjecture 0.1.

The ingredients (i) and (ii) were provided for arbitrary Shimura varieties by Klingler, Ullmo and Yafaev in [KUY16]. Pila, Shankar and Tsimerman [PST21] fulfilled the final missing ingredient (iii), which is achieved by constructing a canonical Weil height h with good functionality properties on the $\overline{\mathbb{Q}}$-points of arbitrary Shimura varieties such that one can bound the heights of special points by relating them to the Faltings height bounds coming from $\mathcal{A}_g$. The construction of h relies heavily on the p-adic Hodge theory.

The reduction step from the height bound of special points to the lower bound of Galois orbits is due to Binyamini, Schmidt and Yafaev [BSY21] based on breakthrough results of Binyamini [Bin22], which greatly improve the Pila-Wilkie estimates for some transcendental varieties (defined by foliations over number fields).

The goal of this seminar is to introduce the André-Oort conjecture and the Pila-Zannier strategy for a general Shimura variety, and to study the proof Conjecture 0.1 in the case of $\mathcal{A}_g$ following [Tsi18] and [BSY21], which can be served as the first step to understand [PST21]. We will focus on the technique of o-minimality.

Talks

Format and Places.

Format: hybrid – in person but will stream simultaneously via Zoom. The speakers are free to choose whether to present in-person or online.

Places: alternating between Frankfurt and Darmstadt (starting from Frankfurt)

- Frankfurt: Room 711 Groß, Robert-Mayer-Straße 10.
- Darmstadt: Room 244 - S2|15, Schlossgartenstraße 7.

What does $\star$, $\star\star$, and $\star\star\star$ mean?

$\star$: Suitable for Masters- or Ph.D. students without much background in complex algebraic geometry; talks should be straightforward to prepare.
$\star\star$: Suitable for Ph.D. students and postdocs; usually requires background in complex algebraic geometry or knowledge of almost all previous talks.
$\star\star\star$: Suitable for ambitious Ph.D. students and postdocs, as well as for Professors. Requires solid background in complex algebraic geometry and/or the willingness to engage with the material in significant depth.

Mathematics is made together.

The goal of the Gaus-AG is not only to learn advances in current research, but also to get to know each other, discuss and participate actively. We strongly encourage you to ask the organizers and your colleagues about any problem arising in the preparation of the talks.

Questions about general state of art and curiosities on the topic of the seminar are more than welcome.

Talk 1: Introduction. (21.04 Frankfurt)[J. Chen]

The organizers will give an overview of the topic of the seminar.

Talk 2: o-minimality I: what is o-minimality? ⋆ (21.04 Frankfurt)[K. Kühn]

Give some motivations for o-minimality, see, e.g., [Cos, Section 1.1 of Chapter 1]. Introduce structures (over $\mathbb{R}$), definable sets and definable maps, and illustrate the basic properties as in [Bak, Propositions 1.1.2 and 1.1.6]. Define o-minimality as "tameness" property. Along the way, introduce the structures $\mathbb{R}_{\text{alg}}$, $\mathbb{R}_{\text{sin}}$, $\mathbb{R}_{\text{exp}}$, $\mathbb{R}_{\text{an}}$, $\mathbb{R}_{\text{an,exp}}$, and state which of them are (not) o-minimal (no proof). Shed some light on the model theoretic perspective [Cos, Theorem 1.13]. References: ([Bak],[Cos], [vdD98])

Talk 3: o-minimality II: definable cell decomposition. ⋆(05.05 Darmstadt)[A. Shavali]

Cover Section 1.2 of [Bak]. More specifically, introduce definable cylindrical cell decompositions of $\mathbb{R}^n$; state the Cell Decomposition Theorem [Bak, Theorem 1.2.6] but prove only the special case given by the Monotonicity Theorem [Bak, Lemma 1.2.8]. Give the definition of *dimension* of a definable set and state some nice properties of definable sets/maps, for example, the uniform finiteness property [vdD98, Lemma 2.13].
I.T.A.: introduce definable topological spaces and provide examples as in [Bak, Subsection 2.1]. References: ([Bak],[Cos], [vdD98])

Talk 4: Application I: o-minimal Chow. ⋆ (05.05 Darmstadt)[Y. Kleibrink]

The goal is to see how the transcendence of an analytic variety can be highly constrained by the tameness of its topology. Recall the difference between analytic and algebraic subvarieties, then recall the classical Chow Theorem (see e.g. [GH78]). Introduce definable complex manifolds and definable complex analytic subsets; state the o-minimal Chow Theorem. A standard reference is [PS08, Sections 3 and 4] and the o-minimal Chow theorem is [PS08, Theorem 5.1]. Provide an "o-minimal" proof of the latter following [Bak, Subsection 1.3]. References: ([Bak],[PS08], [PS09], [PS10])

Talk 5: Application II: Pila-Wilkie counting and the Manin-Mumford conjecture. ⋆⋆⋆ (19.05 Frankfurt)[M. Lara]

The goal is to see how one can recognize/obtain algebraic sets by counting rational points, and then apply it to give a proof of the Manin-Mumford conjecture. State and explain the Pila-Wilkie counting theorem as in [KUY18, Subsection 5.2], then treat it as a blackbox to sketch a proof of the Manin-Mumford conjecture following, e.g., [Sca17, Subsection 5.1]. The whole proof is actually an implementation of the Pila-Zannier strategy in the case of abelian varieties, but the speaker does not need to explain the mechanism in detail in this talk. Instead, (s)he should emphasize the corresponding main ingredients (i),(ii),(iii) (see the introduction part of this program) for abelian varieties, as we are going to generalize them to the case of Shimura varieties in later talks. References: ([PZ08],[Sca17],[KUY18])

Talk 6: Shimura varieties I: a gentle introduction. $\star$ (19.05 Frankfurt)[**A. Thevis**]

The goal is to provide an introduction to Shimura varieties for non-experts. The speaker should avoid to be too technical and provide examples instead of long lists of definitions. A good reference is [Lan16] [1]. Recall the action of $\mathrm{SL}_2(\mathbb{R})$ on the upper-half plane $\mathbb{H}$ and introduce the modular curve $X = \mathrm{SL}_2(\mathbb{Z})\backslash\mathbb{H}$. Explain that it may be rewritten as the double quotient $X = \mathrm{SL}_2(\mathbb{Z})\backslash \mathrm{SL}_2(\mathbb{R})/\mathrm{SO}(2)$; see [Lan16, Example 2.2.7]. A (connected) Shimura variety may be considered as a generalization of such a curve: replacing SL_2 with a semisimple Lie group G, the group $\mathrm{SO}(2)$ with a compact maximal subgroup of G, and $\mathrm{SL}_2(\mathbb{Z})$ with an arithmetic subgroup, so that the quotient G/K is a Hermitian symmetric domain. Provide examples of such symmetric domains in the case of $G = \mathrm{Sp}_{2g}(\mathbb{R})$ following [Lan16, Subsections 3.1.1 & 3.1.2], and of $G = \mathrm{SO}_{n,2}(\mathbb{R})$ following [Lan16, Subsections 3.4.1 & 3.4.2]. State the Baily–Borel Theorem [Lip, Theorem 0.0.0.1]. Give motivation (only ideas): Some moduli spaces are connected Shimura varieties, e.g., $\mathcal{A}_g$ if $G = \mathrm{Sp}_{2g}(\mathbb{R})$ [Lan16, Subsection 3.1.5], [BvdGHZ08, Part 3, Section 10]; moduli spaces of quasi-polarized K3 surfaces if $G = \mathrm{SO}_{n,2}(\mathbb{R})$ [Lan16, Example 5.2.2.11], [GHS07, first two pages]. References: ([Lan16], [Mil17], [BvdGHZ08], [GHS07], [Lip])

Talk 7: o-minimality and Shimura varieties. $\star\star$ (02.06 Darmstadt)[**P. Kiefer**]

The goal is to show that the uniformization map for a Shimura variety is definable in some o-minimal structure if we restrict it to some semi-algebraic fundamental set. Introduce Siegel sets and discuss the reduction theory following [BKT20, Subsetions 2.1-2.2] (see [Bor69] for more information on reduction theory) [2]. Prove the first part of [BKT20, Theorem 1.1 (1)][3]. References: ([BKT20],[Fre20, Section 3], [Bor69],[KUY16])

Talk 8: Shimura varieties II: after Deligne. $\star\star$ (02.06 Darmstadt)[**J. Stix**]

Follow closely [Edix14, Section 7]: give the construction of Shimura varieties à la Deligne [Edix14, Subsections 7.1-7.11]; describe the set of connected components of a Shimura variety in the case when $\boldsymbol{G}^{\mathrm{der}}$ is 1-connected, where $\boldsymbol{G}$ is the defining algebraic group of

[1]For more suggestions and references, please ask Riccardo.
[2]See [Bor19] for an English translation of [Bor69].
[3]The original proof in [KUY16] uses toroidal compactifications which is more sophisticated.

the Shimura variety [Edix14, Subsections 7.12-7.13]; define the action of $\boldsymbol{G}(\mathbb{A}_f)$ on the limit of Shimura varieties taking over all the compact open subgroups of $\boldsymbol{G}(\mathbb{A}_f)$ and define the Hecke correspondence and then define special subvarieties/points of a Shimura variety [Edix14, Subsections 7.16-7.19, 7.23-7.26]; explain why the special points are analytically (in particular, Zariski) dense in any special subvariety [KUY18, Lemma 2.5]; describe the reflex field of a Shimura variety and define the canonical model for zero dimensional Shimura varieties: class field theory (Artin reciprocity) [Edix14, Subsections 7.27-7.28] and state [Edix14, Theorem 7.31]; state the André-Oort conjecture. References: ([Edix14], [Del79],[Del71],[Mil17], [Lan16], [KUY18])

Talk 9: Bi-algebraic geometry: an introduction. ⋆⋆ (23.06 Frankfurt)[**Y. M. Wong**]

Follow closely [KUY18, Section 4]: define bi-algebraic structure as well as its $\overline{\mathbb{Q}}$-enrichment. Focus on its geometric part: give examples and describe their corresponding bi-algebraic subvarieties [KUY18, Subsection 4.1]; define weakly special subvarieties of a Shimura variety [KUY18, Subsection 3.3] and show that they are the same as the bialgebraic ones [KUY18, Theorem 4.9] (see [UY11, Proposition 5.1] for the abelian variety case); introduce the Ax-Lindemann principle and give examples which satisfy this principle [KUY18, Subsection 4.4]. If time permits, briefly explain the proof of this functional transcendence property for Shimura varieties [KUY18, Section 7] — explain how to use Pila-Wilkie counting to prove big monodromy.
I.T.A.: discuss also the arithmetic counterparts [KUY18, Subsection 4.2].

Talk 10: The André-Oort conjecture. ⋆⋆⋆ (23.06 Frankfurt)[**R. Zuffetti**]

The goal is to deduce the André-Oort conjecture for (pure) Shimura varieties by combining the definability result [Talk 7], the functional transcendence result [Talk 9], as well as the Galois lower bounds for special points [BSY21, Theorem 1], using the Pila-Zannier strategy. Illustrate first how the André-Oort conjecture follows from the geometric result [KUY18, Theorem 8.1] (the induction step) and the arithmetic result on lower bounds of Galois orbits of special points [BSY21, Theorem 1], see [KUY18, Section 8] and [Gao16, Section 2]; prove the induction step [KUY18, Theorem 8.1] following [KUY18, Subsection 8.1]. References: ([KUY18], [Gao16], [BSY21])

Talk 11: The André-Oort conjecture for $\mathcal{A}_g$: Galois lower bounds for CM abelian varieties - after Tsimerman and Binyamini-Schmidt-Yafaev. ⋆⋆⋆ (30.06 Darmstadt)[**Y. Li**]

State the height bound conjecture of Binyamini, Schmidt and Yafaev [BSY21, Conjecture 2]. Then particularizing to $\mathcal{A}_g$, illustrate [KUY18, Theorem 8.7] on the heights of pre-special points through the example of the modular curve $Y_0(1)$. Deduce the lower bounds of Galois degrees of CM points by assuming the point-counting theorem [BSY21, Theorem 3] of Binyamini-Schmidt-Yafaev following [BSY21, Section 2]. Prove the aforementioned height bound conjecture for CM points of $\mathcal{A}_g$: introduce the Faltings heights and explain that Faltings heights of CM abelian varieties are discriminant-negligible following [KUY18, Subsections 9.2 & 9.4] (see [FWGSS83, Chapter II] for more information on the Faltings height); state the comparison between the Faltings heights and a Weil height [FWGSS83, Theorem 3.1 of Chapter II].

Talk 12: TBD. (30.06)[**J. Tsimerman**]

References

[Bak] B. Bakker: *Hodge theory and o-minimality*, https://drive.google.com/file/d/1KNXtWekGk1mAyJki-xT6bnbr9ofjdZGs/view

[Bin22] G. Binyamini: *Point counting for foliations over number fields*, Forum of Mathematics, Pi , Volume 10 , 2022 , e6.

[Bor69] A. Borel: *Introduction aux groupes arithmétiques*, (French) Publications de l'Institut de Mathématique de l'Université de Strasbourg, XV. Actualités Scientifiques et Industrielles, No. 1341 Hermann, Paris 1969 125 pp.

[Bor19] A. Borel: *Introduction to arithmetic groups*, Translated from the 1969 French original by Lam Laurent Pham. Edited and with a preface by Dave Witte Morris. University Lecture Series, 73. American Mathematical Society, Providence, RI, 2019. xii+118 pp.

[BKT20] B. Bakker, B. Klingler, J. Tsimerman: *Tame topology of arithmetic quotients and algebraicity of Hodge loci*, J. Amer. Math. Soc. 33 (2020), no. 4, 917–939.

[BSY21] G. Binyamini, H. Schmidt, A. Yafaev: *Lower bounds for Galois orbits of special points on Shimura varieties: a point-counting approach*, arXiv:2104.05842v2.

[BvdGHZ08] J. H. Bruinier, G. van der Geer, G. Harder, D. Zagier, *The 1-2-3 of modular forms*, Lectures from the Summer School on Modular Forms and their Applications held in Nordfjordeid, June 2004, Edited by Kristian Ranestad, Springer-Verlag, Berlin, 2008.

[Cos] M. Coste: *An introduction to o-minimal geometry*, https://perso.univ-rennes1.fr/michel.coste/polyens/OMIN.pdf

[Del71] P. Deligne: *Travaux de Shimura*, (French) Séminaire Bourbaki, 23ème année (1970/71), Exp. No. 389, pp. 123–165. Lecture Notes in Math., Vol. 244, Springer, Berlin, 1971.

[Del79] P. Deligne: *Variétés de Shimura: interprétation modulaire, et techniques de construction de modèles canoniques*, (French) Automorphic forms, representations and L-functions (Proc. Sympos. Pure Math., Oregon State Univ., Corvallis, Ore., 1977), Part 2, pp. 247–289, Proc. Sympos. Pure Math., XXXIII, Amer. Math. Soc., Providence, R.I., 1979.

[Edix14] B. Edixhoven: *Shimura varieties, notes for some introductory lectures*, https://pub.math.leidenuniv.nl/~edixhovensj/talks/2012/2012_06_shimura.pdf

[Fre20] J. Frésan: *Hodge theory and o-minimality*, http://javier.fresan.perso.math.cnrs.fr/o-minimal.pdf

[FWGSS83] G. Faltings, G. Wüstholz, F. Grunewald, N. Schappacher, U. Stuhler: *Rational points*, Second edition. Papers from the seminar held at the Max-Planck-Institut für Mathematik, Bonn/Wuppertal, 1983/1984. Aspects of Mathematics, E6. Friedr. Vieweg & Sohn, Braunschweig, 1986. vi+268 pp. ISBN: 3-528-18593-7

[Gao16] Z. Gao: *About the mixed André-Oort conjecture: reduction to a lower bound for the pure case*, C. R. Math. Acad. Sci. Paris 354 (2016), no. 7, 659–663.

[GH78] P. Griffiths, J. Harris: *Principles of algebraic geometry*, Pure and Applied Mathematics. Wiley-Interscience [John Wiley & Sons], New York, 1978. xii+813 pp.

[GHS07] V. A. Gritsenko, K. Hulek, G. K. Sankaran: *The Kodaira dimension of the moduli of K3 surfaces*, Invent. Math. 169 (2007), no. 3, 519–567.

[KUY16] B. Klingler, E. Ullmo, A. Yafaev: *The hyperbolic Ax-Lindemann-Weierstrass conjecture*, Publ. Math. Inst. Hautes Études Sci. 123 (2016), 333-360.

[KUY18] B. Klingler, E. Ullmo, A. Yafaev: *Bi-algebraic geometry and the André-Oort conjecture*, Algebraic geometry: Salt Lake City 2015, 319-359, Proc. Sympos. Pure Math., 97.2, Amer. Math. Soc., Providence, RI, 2018.

[Lan16] K. W. Lan: *An example-based introduction to Shimura varieties*, https://www-users.cse.umn.edu/~kwlan/articles/intro-sh-ex.pdf.

[Lip] M. Lipnowsky, *Baily-Borel compactification*, https://drive.google.com/file/d/1iUTbJAcHSI6J7GFzmtDRD7PwPSI73csU/view.

[Mil17] J. S. Milne: *Introduction to Shimura varieties*, https://www.jmilne.org/math/xnotes/svi.pdf.

[PS08] Y. Peterzil and S. Starchenko: *Complex analytic geometry in a nonstandard setting*, In: Model theory with applications to algebra and analysis. Vol.1. Vol. 349. London Math. Soc. Lecture Note Ser. Cambridge Univ. Press, Cambridge, 2008.

[PS09] Y. Peterzil and S. Starchenko: *Complex analytic geometry and analytic-geometric categories*, J. Reine Angew. Math. 626 (2009), 39–74.

[PS10] Y. Peterzil and S. Starchenko: *Tame complex analysis and o-minimality*, Proceedings of the International Congress of Mathematicians. Volume II, 58–81, Hindustan Book Agency, New Delhi, 2010.

[PST21] J. Pila, A. N. Shankar, J. Tsimerman: *Canonical Heights on Shimura Varieties and the André-Oort conjecture*, with an appendix by H. Esnault and M. Groechenig, arXiv:2109.08788v2.

[PZ08] J. Pila, U. Zannier: *Rational points in periodic analytic sets and the Manin-Mumford conjecture*, Atti Accad. Naz. Lincei Rend. Lincei Mat. Appl. 19 (2008), no. 2, 149–162.

[Sca17] T. Scanlon: *O-minimality as an approach to the André-Oort conjecture*, Around the Zilber-Pink conjecture/Autour de la conjecture de Zilber-Pink, 111–165, Panor. Synthèses, 52, Soc. Math. France, Paris, 2017.

[Tsi18] J. Tsimerman: *The André-Oort conjecture for $\mathcal{A}_g$*, Ann. of Math. (2) 187 (2018), no. 2, 379-390.

[UY11] E. Ullmo, A. Yafaev: *A characterization of special subvarieties*, Mathematika 57 (2011), no. 2, 263–273.

[vdD98] L. van den Dries: *Tame topology and o-minimal structures*, London Mathematical Society Lecture Note Series, 248. Cambridge University Press, Cambridge, 1998. x+180 pp.