Workshop: Higher Algebra

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Organization

Each of the talks will be **90 minutes** long. After the last talk on friday there is a hike planned for all who want to participate. We will finish the hike at 18:30 at the "Restaurant Bölle" where everybody is welcome to join us for a dinner.

	Wednesday 16.03.	Thursday 17.03.
9:00 - 10:30	Talk 1	Talk 5
10:30 - 11:00	Coffeebreak	Coffeebreak
11:00 - 12:30	Talk 2	Talk 6
12:30 - 14:00	Lunch	Lunch
14:00 - 15:30	Talk 3	Talk 7
15:30 - 16:00	Coffeebreak	Coffeebreak
16:00 - 17:30	Talk 4	Talk 8

	Friday 17.03.
9:00 - 10:30	Talk 9
10:30 - 11:00	Coffeebreak
11:00 - 12:30	Talk 10
12:30 - 13:30	Lunch
13:30 - 15:00	Talk 11
15:00 - 16:00	
16:00 - 18:30	Hike
18:30	Dinner

Introduction

The aim of this workshop is to understand the fundamentals of derived algebraic geometry. One aspect is the study of the derived category over a ring. One can show that there are complexes that are Zariski locally zero but not globally. In particular, the derived category in the classical world of algebraic geometry does not satisfy descent. The problem comes from the fact that we only consider morphisms up to quasi-isomorphism. This makes gluing at first impossible since we have to quotient out a relation. But instead of viewing morphisms up to some relation, we can add the relation in a nice way as a datum to our category and doing this iteratively for the relations naturally leads to the notion of the derived ∞-category. It turns out that the derived ∞-category satisfies descent for the fpqc topology.

To see this one has to realize the derived ∞-category as a category of modules over E_∞-rings. Then it seems plausible that modules satisfy descent, as for quasi-coherent modules over schemes. But as it turns out E_∞-rings are not the right objects when one wants to do algebraic geometry in characteristic other than 0. Instead one wants to work with animated rings (also called simplicial commutative rings). These are often used to transport results in the smooth setting to the non-smooth setting. The idea is to write every animated ring as a colimit of smooth rings and extend functors in this way. There is also a way to connect animated rings and E_∞-rings which allows us to define modules over those which in special cases give us the derived ∞-category.

One can also try to do algebraic geometry on animated rings, which we call derived algebraic geometry. Doing this, naturally gives objects from the classical world that are interesting in algebraic geometry, like the cotangent complex. This is for example used to extend theories of smooth schemes, such as algebraic cobordism, to the world of lci schemes.

Structure

Even though we can not fully cover the theory of derived algebraic geometry, we at least want to learn the most important aspects. The world of ∞-categories and model categories are highly linked (one analog would be manifolds and local coordinates). So, we will first look closely at model categories and their relation to ∞-categories. This allows us to understand the ∞-category of spectra, which plays a very important role since we can attach a symmetric monoidal structure to spectra such that the ring objects are E_∞-rings. We are then ready to look at modules over E_∞-rings which we can identify in special cases with the derived ∞-category. This allows us to finally do derived commutative algebra, i.e. study animated rings and morphisms between them. Finally, we finish the workshop with descent for the derived ∞-category.

Model categories

1. The main source of this talk is [DS95]. Define model categories and explain the model structures on the category of topological spaces **TOP** (see [DS95, Ex. 3.5]) and the category of chain complexes **CH**. Analyse fibrant and cofibrant objects of **TOP** and **CH**. Relate fibrations and cofibrations with liftings of the other. Define the homotopy category of a model category and show that it is the same as the localization with respect to weak equivalences. Further make it explicit by looking at the model category of chain complexes. Define derived functors and relate them to the classical theory of derived functors in the derived category.

2. Consider simplicial model categories and show how to get an ∞-category from a simplicial model category as explained on the first few lines of [Lur09, A.2. (2)]. Define the homotopy limit/colimit of model categories and give the loop/suspension functor as an example in **TOP** and **CH**. Mention that the ∞-limit/colimit in the associated ∞-category is computed upto homotopy by the homotopy limit/colimit. Show that homotopy limits are not limits in the homotopy category. Define Quillen adjunctions/equivalences and show that Quillen adjunctions induce adjunctions on the ∞-level and in good cases a zigzag of Quillen equivalences is the same as an equivalence on the underlying ∞-categories.

Stable ∞-categories

3. The aim of this talk is to explain [Lur17, §1.1, 1.2]. Define stable ∞-categories and show that the homotopy category of ∞-category is triangulated (only prove that it is additive and explain how distinguished triangles are defined). State [Lur17, 1.1.3.1 - 1.1.3.4] (and maybe prove [Lur17, Prop. 1.1.3.4] if time permits). Define t-structures and homotopy groups.

4. This talk will look closely at the derived ∞-category and the ∞-category of spectra (see [Lur17, §1.3, §1.4]). Briefly mention how a differential graded category gives rise to an ∞-category. Apply this to construct the derived ∞-category. Define this directly

for Grothendieck abelian categories. Show that it is stable. Define the ∞-category associated to a model category and mention how the derived ∞-category is the ∞-category associated to the model category of chain complexes [Lur17, Prop. 1.3.5.15]. Construct spectrum objects and particularly mention [Lur17, Prop. 1.4.4.4] (here one needs to define presentable ∞-categories but this should not take to much time).

Symmetric monoidal ∞-categories

5. Define ∞-operads [Lur17, §2.1] and algebra objects and show how in the case of classical colored operads the notion of an algebra object for the monoidal category of abelian groups recovers the classical notion of a ring. Explain how the underlying ∞-category of an ∞-operad can be seen as a monoidal category [Lur17, Rem. 2.1.2.20]. Explain the rectification argument connecting module/algebra objects on monoidal ∞-categories with their model categorical counterpart (see introduction of [Lur17, §4.1, 4.5]) and generally the connection between monoidal model categories and symmetric monoidal ∞-categories. Define monoidal functors and show some basic properties like limits/colimits, forgetful functors of modules and algebra objects as explained in the introduction of [Lur17, §3.2, 3.4].

Spectra

6. Define the model category of spectra using the model structure as Kan complexes and pointed topological spaces (see [HSS98] and [Lur17, §1.4.3]). Look at modules and rings over it. Define stable homotopy groups. Go to the ∞-categorical world and show how to define a t-structure in spectra [Lur17, §1.4.3]. Finally mention the symmetric monoidal equivalence between the modules over discrete spectral rings and the derived ∞-category (see [Lur17, Thm. 7.1.2.13]).

Modules

7. This talk is based on [Lur17, §7.1, 7.2]. Look at properties of modules over spectra (flat/projective/finite projective). Define the t-structure, show that the heart of this t-structure is (nerve of) the category of modules. Explain the Tor-spectral sequence and how one can compute it. Sketch Lazard's Theorem and show that projective spectral modules over discrete ring are the same as projective modules over the underlying π_0 of that ring. Define perfect modules and prove that they are precisely the dualizable ones (and thus this notion agrees with the one on the derived category).

Animated rings

8. Explain the animation of a category as in [CS21, 5.1.4] and give examples via rings and modules. Bring them into connection with the constructions given in [Lur18, §25.1, §25.2.1]. In particular mention [Lur18, Rem. 25.1.1.3]. Show how to define homotopy groups of animated rings and mention that they are the same as the ones given by the underlying simplicial groups (which are Kan complexes). Compare animated rings and DG-rings (see for example [Ric08]) and DG-rings and E_∞-rings (see [Lur17, §7.4.1]).

9. Define properties of animated rings (étale/smooth/l.f.p) (see [TV08, Def. 2.4.1.3], we omit the word strong here) and explain the localization of an animated ring with respect to an element (see [TV08, Prop. 1.2.9.1]). Define the cotangent complex as in [Lur18, §25.2] and the relative cotangent complex.

10. Relate the cotangent complex to the classical one and state the connection between the cotangent complex of animated rings and their underlying E_∞ counterpart (see [Lur18, §25.3.5]). State and prove [TV08, Thm. 2.2.2.6] (only prove the smooth part).

∞-Descent

11. Explain Grothendieck topologies as in [Lur18, §A.3.1,A.3.2] and [Lur09, §6.2.2]. Give examples of Grothendieck topologies such as the étale and fpqc topology (see [Lur18, §B.6.1]). Define the notion of Čech descent (see [Lur18, §A.3.3]) and show that for an animated ring A, the functor $\mathrm{Hom}_{\mathrm{Ani(Ring)}}(A, -)$ satisfies fpqc descent (use [Lur18, §D.6.3]) and sketch that animated modules satisfy fpqc descent (see [Lur18, §D.3.5]) - in particular the derived ∞-category satisfies fpqc descent (which is not true for its homotopy category).

References

[CS21] Kestutis Cesnavicius and Peter Scholze. Purity for flat cohomology, 2021. `arXiv:1912.10932`.

[DS95] W. G. Dwyer and J. Spaliński. Homotopy theories and model categories. In *Handbook of algebraic topology*, pages 73–126. North-Holland, Amsterdam, 1995. `doi:10.1016/B978-044481779-2/50003-1`.

[HSS98] Mark Hovey, Brooke Shipley, and Jeff Smith. Symmetric spectra, 1998. `arXiv:9801077`.

[Lur09] Jacob Lurie. *Higher topos theory*, volume 170 of *Annals of Mathematics Studies*. Princeton University Press, Princeton, NJ, 2009. `doi:10.1515/9781400830558`.

[Lur17] Jacob Lurie. Higher algebra. `https://www.math.ias.edu/~lurie/papers/HA.pdf`, 2017.

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[Ric08] Birgit Richter. Divided power structures and chain complexes, 2008. `arXiv:0811.4704`.

[TV08] Bertrand Toën and Gabriele Vezzosi. Homotopical algebraic geometry. II. Geometric stacks and applications. *Mem. Amer. Math. Soc.*, 193(902):x+224, 2008. `doi:10.1090/memo/0902`.