Workshop: Higher Algebra

Can Yaylali

November 5, 2021

1 Model categories

1. The main source of this talk is [DS95]. Define model categories and explain the model structures on the category of topological spaces **TOP** (see [DS95, Ex. 3.5]) and the category of chain complexes **CH**. Analyse fibrant and cofibrant objects of **TOP** and **CH**. Relate fibrations and cofibrations with liftings of the other. Define the homotopy category of a model category and show that it is the same as the localization with respect to weak equivalences. Further make it explicit by looking at the model category of chain complexes. Define derived functors and relate them to the classical theory of derived functors in the derived category.

2. Consider simplicial model categories and show how to get an ∞-category from a simplicial model category as explained on the first few lines of [Lur09, A.2. (2)]. Define the homotopy limit/colimit of model categories and give the loop/suspension functor as an example in **TOP** and **CH**. Mention that the ∞-limit/colimit in the associated ∞-category is computed upto homotopy by the homotopy limit/colimit. Show that homotopy limits are not limits in the homotopy category. Define Quillen adjunctions/equivalences and show that Quillen adjunctions induce adjuctions on the ∞-level and in good cases a zigzag of Quillen equivalences is the same as an equivalence on the underlying ∞-categories.

2 Stable ∞-categories

1. The aim of this talk is to explain [Lur17, §1.1, 1.2]. Define stable ∞-categories and show that the homotopy category of ∞-category is triangulated (only prove that it is additive and explain how distinguished

triangles are defined). State [Lur17, 1.1.3.1 - 1.1.3.4] (and maybe prove [Lur17, Prop. 1.1.3.4] if time permits). Define t-structures and homotopy groups.

2. This talk will look closely at the derived ∞-category and the ∞-category of spectra (see [Lur17, §1.3, §1.4]). Briefly mention how a differential graded category gives rise to an ∞-category. Apply this to construct the derived ∞-category. Define this directly for Grothendieck abelian categories. Show that it is stable. Define the ∞-category associated to a model category and mention how the derived ∞-category is the ∞-category associated to the model category of chain complexes [Lur17, Prop. 1.3.5.15]. Construct spectrum objects and particularly mention [Lur17, Prop. 1.4.4.4] (here one needs to define presentable ∞-categories but this should not take to much time).

3 Symmetric monoidal ∞-categories

Define ∞-operads [Lur17, §2.1] and algebra objects and show how in the case of classical colored operads the notion of an algebra object for the monoidal category of abelian groups recovers the classical notion of a ring. Explain how the underlying ∞-category of an ∞-operad can be seen as a monoidal category [Lur17, Rem. 2.1.2.20]. Explain the rectification argument connecting module/algebra objects on monoidal ∞-categories with their model categorical counterpart (see introduction of [Lur17, §4.1, 4.5]) and generally the connection between monoidal model categories and symmetric monoidal ∞-categories. Define monoidal functors and show some basic properties like limits/colimits, forgetful functors of modules and algebra objects as explained in the introduction of [Lur17, §3.2, 3.4].

4 Spectra

Define the model category of spectra using the model structure as Kan complexes and pointed topological spaces (see [HSS98] and [Lur17, §1.4.3]). Look at modules and rings over it. Define stable homotopy groups. Go to the ∞-categorical world and show how to define a t-structure in spectra [Lur17, §1.4.3]. Finally mention the symmetric monoidal equivalence between the modules over discrete spectral rings and the derived ∞-category (see [Lur17, Thm. 7.1.2.13]).

5 Modules

This talk is based on [Lur17, §7.1, 7.2]. Look at properties of modules over spectra (flat/projective/finite projective). Define the t-structure, show that the heart of this t-structure is (nerve of) the category of modules. Explain the Tor-spectral sequence and how one can compute it. Sketch Lazard's Theorem and show that projective spectral modules over discrete ring are the same as projective modules over the underlying π_0 of that ring. Define perfect modules and prove that they are precisely the dualisable ones (and thus this notion agrees with the one on the derived category).

6 Animated rings

1. Explain the animation of a category as in [CS21, 5.1.4] and give examples via rings and modules. Bring them into connection with the constructions given in [Lur18, §25.1, §25.2.1]. In particular mention [Lur18, Rem. 25.1.1.3]. Show how to define homotopy groups of animated rings and mention that they are the same as the ones given by the underlying simplicial groups (which are Kan complexes). Compare animated rings and DG-rings (see for example [Ric08]) and DG-rings and E_∞-rings (see [Lur17, §7.4.1]).

2. Define properties of animated rings (étale/smooth/l.f.p) (see [TV08, Def. 2.4.1.3], we omit the word strong here) and explain the localization of an animated ring with respect to an element (see [TV08, Prop. 1.2.9.1]). Define the cotangent complex as in [Lur18, §25.2] and the relative cotangent complex.

3. Relate the cotangent complex to the classical one and state the connection between the cotangent complex of animated rings and their underlying E_∞ counterpart (see [Lur18, §25.3.5]). State and prove [TV08, Thm. 2.2.2.6] (only prove the smooth part).

7 ∞-Descent

Explain Grothendieck topologies as in [Lur18, §A.3.1,A.3.2] and [Lur09, §6.2.2]. Give examples of Grothendieck topologies such as the étale and fpqc topology (see [Lur18, §B.6.1]). Define the notion of Čech descent (see [Lur18, §A.3.3]) and show that for an animated ring A, the functor $\mathrm{Hom}_{\mathrm{Ani(Ring)}}(A, -)$ satisfies fpqc descent (use [Lur18, §D.6.3]) and sketch that animated modules satisfy fpqc descent (see [Lur18, §D.3.5]) - in particular the derived ∞-category satisfies fpqc descent (which is not true for its homotopy category).

References

[CS21] Kestutis Cesnavicius and Peter Scholze. Purity for flat cohomology, 2021. `arXiv:1912.10932`.

[DS95] W. G. Dwyer and J. Spaliński. Homotopy theories and model categories. In *Handbook of algebraic topology*, pages 73–126. North-Holland, Amsterdam, 1995. `doi:10.1016/B978-044481779-2/50003-1`.

[HSS98] Mark Hovey, Brooke Shipley, and Jeff Smith. Symmetric spectra, 1998. `arXiv:9801077`.

[Lur09] Jacob Lurie. *Higher topos theory*, volume 170 of *Annals of Mathematics Studies*. Princeton University Press, Princeton, NJ, 2009. `doi:10.1515/9781400830558`.

[Lur17] Jacob Lurie. Higher algebra. `https://www.math.ias.edu/~lurie/papers/HA.pdf`, 2017.

[Lur18] Jacob Lurie. Spectral algebraic geometry. `https://www.math.ias.edu/~lurie/papers/SAG-rootfile.pdf`, 2018.

[Ric08] Birgit Richter. Divided power structures and chain complexes, 2008. `arXiv:0811.4704`.

[TV08] Bertrand Toën and Gabriele Vezzosi. Homotopical algebraic geometry. II. Geometric stacks and applications. *Mem. Amer. Math. Soc.*, 193(902):x+224, 2008. `doi:10.1090/memo/0902`.