Hodge theory of matroids

GAUS-Seminar – WiSe 2021/22

Organizer: Martin Ulirsch

List of talks

Matroids (as introduced by Whitney) are a common generalization of finite graphs and of finite sets of vectors in a vector space. They are characterized by having several, seemingly quite different (so-called "cryptomorphic") definitions. In the case of a finite set of vectors in a vector space, a matroid for example encodes which subsets of these vectors are linearly independent. In the case of a finite graph, it encodes the simple cycles of the graph. It is expected, however, that 100% of matroids are not of this type.

The theory of matroids has seen a recent surge of interest, most notably in the groundbreaking results of Adiprasito, Huh, and Katz [AHK18], where they prove the so-called log-concavity conjecture of Rota and Welsh about the coefficients of the reduced characteristic polynomial of a matroid. In order to prove this statement, the authors of [AHK18] develop a new version of Hodge theory for matroids that includes an analogue of the Hodge-Riemann relations and of the Hard Lefschetz Theorem, the so-called "Hodge package".

Beyond the classical results of the Hodge theory of compact Kähler manifolds, which inspire this whole story, there are several incarnations of similar "Hodge packages", for example for the ring of algebraic cycles modulo homological equivalence on a smooth projective variety, for McMullen's polytope algebra, for Karu's combinatorial intersection cohomology of convex polytopes, and for the reduced Soergel bimodule of a Coxeter group.

The goal of this seminar is to introduce ourselves to the (probably unfamiliar) notion of a matroid, to explore its connections to complex algebraic geometry (in particular to Hodge theory), and to give an overview of the Hodge theory of matroids developed in [AHK18] and its application to the log-concavity conjecture.

A good and concise survey of the topics covered in this seminar is [Bak18], which can be complemented by [AHK17, Huh18a, Huh18b]. Matroid theory, aimed at algebraic geometers, is surveyed in [Kat16] and there are several textbooks, including [Oxl92] and [Wel76].

Background knowledge and dependence of the talks

Talk 2-5 require essentially no advanced background. Basic linear algebra (and maybe a bit of algebra) as well as the willingness to learn about new concepts is all that is required. Talk 6, a priori, also only requires the same amount of background, but it can profit from giving some algebraic-geometric motivation (in particular, from the construction of the Grothendieck ring of varieties). Talk 7, 8, and 10 require a certain amount of background in complex algebraic geometry (intersection theory, Hodge theory, toric varieties), as indicated in the talk descriptions. Talk 9, 11, and 12 can in principle be approached without background in algebraic geometry. Talk 11 and 12 require the most work, since the speaker will have to get into the details of [AHK18].

Usually talks happening on the same day have a common theme. Speakers are asked to communicate to make sure their talks "talk to each other".

What does $*$, $**$, and $***$ mean?

$*$: Suitable for Masters- or Ph.D. students without much background in complex algebraic geometry; talks should be straightforward to prepare.
$**$: Suitable for Ph.D. students and postdocs; usually requires background in complex algebraic geometry or knowledge of almost all previous talks.
$***$: Suitable for ambitious Ph.D. students and Postdocs, as well as for Professors. Requires solid background in complex algebraic geometry and/or the willingness to engage with the material in significant depth.

Virtual or not?

The seminar will commence virtually via Zoom and the first three sessions (up to Talk 6) will definitely only take place virtually. If the general pandemic situation allows for it, we might be able to have one or two sessions in the end as in-person events.

Talk 1: Overview (28.10.) [by default: Martin U.]

The organizer will give an overview of the topic of the seminar.

Talk 2: Matroid basics I – cryptomorphisms and examples (28.10) $*$ [Ingmar Metzler]

Introduce the different cryptomorphic axiom systems for matroids as in [Wel76, Section 2.2] (in particular their definition in terms of "independent sets" and Theorem 1-5 without proofs). Then give some central examples of matroids (uniform matroids, vectorial matroids, graphic and cographic matroids), as in [Wel76, Section 2.3] and [Kat16, Section 4]. Mention what it means for a matroid to be realizable [Kat16, Def. 3.3].

Talk 3: Matroid basics II – exercise session (11.11.) [no speaker]

Exercise session where participants will be split in breakout rooms and are asked to work in groups on a selection of elementary exercises to familiarize themselves with matroids.

Talk 4: Matroid basics III – the lattice of flats (11.11.) $*$ [Arne Kuhrs]

Introduce the lattice of flats, as e.g. in [Oxl92, Section 1.7] and explain how it forms yet another cryptomorphic definition of a simple matroid. The central result is [Oxl92, Theorem 1.7.5] and, ideally, a full proof should be given. You will need to introduce some terminology first: simple matroids, posets, and (geometric) lattices (also see [Bak18, Section 3.1] for a quick overview).

Talk 5: Operations on matroids (25.11.) $*$ [Pedro Souza]

Introduce the basic operations on matroids, as e.g. summarized in [Kat16, Section 5]. The most important ones for the rest of the seminar are deletion/contraction, direct sums, and duality (that correspond to [Kat16, Section 5.1, 5.2, and 5.3]). Mention the further operations from [Kat16, Section 5.4-5.7] only if there is time to do so.

Talk 6: The characteristic polynomial (25.11.) $*$ or $**$ [Lucie Devey]

Introduce the characteristic polynomial of a matroid, as in [Kat16, Section 7.1 and 7.3]. Finish by stating the Rota-Welsh log-concavity conjecture from [Kat16, Section 7.5]. If there is time, you can include the motivic characterization of the characteristic polynomial in [Kat16, Section 7.2] for motivation (for which you would have to know about the Grothendieck ring of varieties). The same material is also surveyed in [Bak18, Section 3].

Talk 7: A crash course on toric varieties (9.12.) $**$ [Felix Goebler]

Give a crash course on the basic theory of toric varieties, in particular how to construct them from a rational polyhedral fan, as e.g. in [Kat16, Section 10.1]. Since this is only a 60-minute talk, it might be a good idea to mostly focus on examples. This talk is perfect for someone, who has already encountered toric varieties before and wants to share this story with everyone else. It requires no knowledge of the previous talks.

Talk 8: Minkowski weights and the Chow ring of a toric variety (9.12) $**$ [Luca Battistella]

Start by recalling the basic notions of intersection theory and explain how to think of of the Chow cohomology ring of a smooth and complete toric variety in terms of Minkowski weights, as outlined in [Kat16, Section 10.2]. If there is time, say something about how to prove this identification (see [FS97]). This talk is particularly suitable for a speaker who has already seen some basic notions of intersection theory and requires no knowledge of any of the previous talks but Talk 7.

Talk 9: Bergman fans and the permutohedral variety (20.1.) $**$ [Stefan Rettenmayr]

Introduce the Bergman fan of a matroid and explain how it allows us to think of a matroid as a Minkowski weight on the permutohedral fan. This can be carried out as in [Kat16, Section 11]; but also [Huh18b] can help. The purpose of this talk is to connect Talks 2-6 with Talk 7-8.

Talk 10: Intersection theory on the permutohedral variety (20.1.) $***$ [Andreas Gross]

Prove the log-concavity conjecture for realizable matroids using standard (but non-trivial) methods from algebraic geometry (as outlined in the proof of [Kat16, Theorem 12.2]). The wonderful compactification of a hyperplane arrangement will have a surprise appearance here. Also compare this to [Huh18b] and the original article [HK12]. This talk requires a fairly strong background in complex algebraic geometry.

Talk 11: The Chow ring of a matroid (3.2.) $***$ [Alex Küronya]

Define the Chow ring of an arbitrary matroid, as in [AHK18, Section 4 and 5], state the Hodge-Riemann bilinear relations and the Hard Lefschetz Theorem, and give an overview how to deduce the general log-concavity conjecture from these results (as in [AHK18, Section 9]).

Talk 12 by June Huh (Princeton): Kazhdan-Lusztig theory of matroids and its relation to Hodge theory (3.2)

The foremost expert on the topic of our seminar will give a virtual talk on the big picture and the next chapter of this story.

Last meeting: Discussion of topics for the next seminar (17.2)

References

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[AHK18] ———, *Hodge theory for combinatorial geometries*, Ann. of Math. (2) **188** (2018), no. 2, 381–452.

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[HK12] June Huh and Eric Katz, *Log-concavity of characteristic polynomials and the Bergman fan of matroids*, Math. Ann. **354** (2012), no. 3, 1103–1116.

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[Huh18b] ———, *Tropical geometry of matroids*, Current developments in mathematics 2016, Int. Press, Somerville, MA, 2018, pp. 1–46.

[Kat16] Eric Katz, *Matroid theory for algebraic geometers*, Nonarchimedean and tropical geometry, Simons Symp., Springer, [Cham], 2016, pp. 435–517.

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[Wel76] D. J. A. Welsh, *Matroid theory*, L. M. S. Monographs, No. 8, Academic Press [Harcourt Brace Jovanovich, Publishers], London-New York, 1976.