Root Data

§1 Definition and First Observations

<u>Def. 1.1</u> A root datum is a quadruple $\Psi = (X, R, X^\vee, R^\vee)$ where

- $X, X^\vee$ free abelian groups of finite rank endowed with a perfect pairing $\langle\, , \rangle : X \times X^\vee \longrightarrow \mathbb{Z}$ (i.e., $\langle\, , \rangle$ identifies each of $X, X^\vee$ with the dual of the other)
- $R \subseteq X$, $R^\vee \subseteq X^\vee$ finite subsets, in bijection via a map
$$R \longrightarrow R^\vee, \quad \alpha \longmapsto \alpha^\vee$$

subject to the following conditions:

For $\alpha \in R$ define endomorphism

$$s_\alpha : X \longrightarrow X, \qquad s_\alpha^\vee : X^\vee \longrightarrow X^\vee,$$
$$s_\alpha(x) = x - \langle x, \alpha^\vee \rangle \alpha \qquad s_\alpha^\vee(y) = y - \langle \alpha, y \rangle \alpha^\vee$$

We demand for all $\alpha \in R$

(RD1) $\langle \alpha, \alpha^\vee \rangle = 2$,

(RD2) $s_\alpha(R) \subseteq R$ and $s_\alpha^\vee(R^\vee) \subseteq R^\vee$.

<u>Obs. 1.2</u> (RD1) implies $s_\alpha^2 = \mathrm{id}_X$ and $s_\alpha(\alpha) = -\alpha$ for $\alpha \in R$. In particular, $s_\alpha \in \mathrm{Aut}(X)$ and s_α acts on R by permutation.

proof. $s_\alpha(s_\alpha(x)) = s_\alpha(x) - \langle s_\alpha(x), \alpha^\vee \rangle \alpha$
$$= x - \langle x, \alpha^\vee \rangle \alpha - \langle x, \alpha^\vee \rangle \alpha + \langle x, \alpha^\vee \rangle \overbrace{\langle \alpha, \alpha^\vee \rangle}^{=2} = x,$$
$$s_\alpha(\alpha) = \alpha - \overbrace{\langle \alpha, \alpha^\vee \rangle}^{=2} \alpha = -\alpha.$$

Obs 1.3 To every root datum $\Psi = (X, R, X^\vee, R^\vee)$ there is its dual root datum $\Psi^\vee = (X^\vee, R^\vee, X, R)$.

Examples 1.4 (i) Endow $X = X^\vee = \mathbb{Z}^n$ with the standard pairing and define

$$R = R^\vee = \{ e_i - e_j \in \mathbb{Z}^n \mid i \neq j \},$$

where $e_i \in \mathbb{Z}^n$ denotes the standard basis vectors.

(ii) Again consider $X = X^\vee = \mathbb{Z}^n$. Now define

$$R = \{ \pm 2e_i, \ \pm e_i \pm e_j \mid i \neq j \},$$

$$R^\vee = \{ \pm e_i, \ \pm e_i \pm e_j \mid i \neq j \}$$

and a bijection $\pm 2e_i \longmapsto \pm e_i$, $\pm e_i \pm e_j \longmapsto \pm e_i \pm e_j$.

§2 Weyl Group

Def 2.1 Define the Weyl group to be

$$W = W(\Phi) = \langle s_\alpha \mid \alpha \in R \rangle \subseteq \mathrm{Aut}(X).$$

Lemma 2.2 The Weyl group W is finite.

proof. Define the $\mathbb{R}$-vector spaces

$$V = \mathbb{R} \otimes_{\mathbb{Z}} X, \qquad V^\vee = \mathbb{R} \otimes_{\mathbb{Z}} X^\vee.$$

The perfect pairing $\langle\, , \rangle : X \times X^\vee \longrightarrow \mathbb{Z}$ extends to a perfect pairing $\langle\, , \rangle : V \times V^\vee \longrightarrow \mathbb{R}$.

We regard R resp $R^\vee$ as subsets of V resp. $V^\vee$ and the Weyl group as subgroup $W \subseteq \mathrm{Aut}_{\mathbb{R}}(V)$ via the embedding

$$\mathrm{Aut}_{\mathbb{Z}}(X) \longrightarrow \mathrm{Aut}_{\mathbb{R}}(V), \quad \varphi \longmapsto \mathrm{id}_{\mathbb{R}} \otimes \varphi.$$

Denote by $U = \mathrm{span}_{\mathbb{R}}(R) \subseteq V$ (resp. $U^\vee = \mathrm{span}_{\mathbb{R}}(R^\vee) \subseteq V^\vee$) the subspace of V (resp. $V^\vee$) spanned by R (resp. $R^\vee$).

Consider the $\mathbb{R}$-linear map

$$f : V \longrightarrow V^\vee, \qquad f(x) = \sum_{\alpha \in R} \langle x, \alpha^\vee \rangle \alpha^\vee.$$

Using that $s_\alpha^\vee$ permutes $R^\vee$, we obtain

$$f(\alpha) = \tfrac{1}{2} \langle \alpha, f(\alpha) \rangle \alpha^\vee \qquad (\alpha \in R).$$

From

$$\langle x, f(x) \rangle = \sum_{\alpha \in R} \langle x, \alpha^\vee \rangle \langle x, \alpha^\vee \rangle = \sum_{\alpha \in R} \langle x, \alpha^\vee \rangle^2 \qquad (*)$$

it follows $0 < \langle \alpha, f(\alpha) \rangle$. Consequently, $\mathrm{im}(f) = U^\vee = f(U)$ and hence $V = U + \ker(f)$. Moreover, $(*)$ implies

$$\ker(f) = \{ x \in V \mid \langle x, \alpha^\vee \rangle = 0 \text{ for all } \alpha \in R \}.$$

We conclude that s_α acts on $\ker(f)$ as the identity and on U by permuting R. The assertion transfers to $W = \langle s_\alpha \mid \alpha \in R \rangle$. Thus, $\varphi \in W$ is completely determined by its permutation of R. As there are only finitely many permutations, W is finite. $\square$

<u>Def 2.3</u> Given an Euclidean vector space $(E, (\,,\,))$ a root system Φ is a finite set of non-zero vectors satisfying

- Φ spans E,
- For every $\alpha \in \Phi$, the set Φ is closed under reflections $x \longmapsto x - \frac{2(x,\alpha)}{(\alpha,\alpha)}\alpha$ through the hyperplane perpendicular to α,
- For $\alpha, \beta \in \Phi$, the number $\frac{2(\alpha,\beta)}{(\alpha,\alpha)}$ is an integer.

<u>Obs 2.4</u> Given a root datum $\Psi = (X, R, X^\vee, R^\vee)$ let $U \subseteq \mathbb{R} \otimes_{\mathbb{Z}} X$ the subspace generated by R (as in the proof of Lemma 1.5). Pick a scalar product $(\,,\,)$ on U that is invariant under the Weyl group. (Such an invariant scalar product can be found, since the Weyl group is finite.)
Then R is a root system in U.
The automorphisms on U induced by s_α $(\alpha \in R)$ are the reflections in $(U, (\,,\,))$.

§3 System of positive roots & reduced root datum

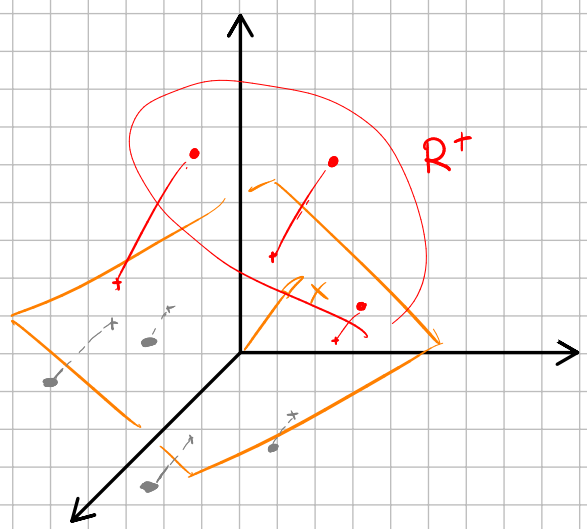
Def 3.1 We call a root datum $\Psi = (X, R, X^\vee, R^\vee)$ reduced if $2\alpha \notin R$ for any root $\alpha \in R$.

Def 3.2 Let $W = W(\Psi)$ be the Weyl group and $(\ ,\ )$ a W-inv. scalar product on $V = \mathbb{R} \otimes_{\mathbb{Z}} X$. We call $R^+ \subseteq R$ a system of positive roots if there ex. $x \in V$ s.t. $(\alpha, x) \neq 0$ for all $\alpha \in R$ and

$$R^+ = \{\alpha \in R \mid (\alpha, x) > 0\}$$

(or equivalently, if there ex. $\lambda \in X^\vee$ s.t. $\langle \alpha, \lambda \rangle \neq 0$ for all $\alpha \in R$ and

$$R^+ = \{\alpha \in R \mid \langle \alpha, \lambda \rangle > 0\}.)$$



Obs 3.3 (1) The convex hull of R^+ in V does not contain 0.

(2) R is the disjoint union of R^+ and $-R^+$.

(3) If $\alpha, \beta \in R^+$ s.t. $\alpha + \beta \in R$ then $\alpha + \beta \in R^+$.

(4) $(R^+)^\vee$ is a system of positive roots in $R^\vee$, i.e., there is an $x \in X$ s.t. $\langle x, \alpha^\vee \rangle \neq 0$ for all $\alpha \in R$ and

$$(R^+)^\vee = \{\alpha^\vee \in R^\vee \mid \langle x, \alpha^\vee \rangle > 0\}.$$

§ 4 Two roots

Lemma 4.1 Let $\alpha, \beta \in R$ be lin. indep. roots

(i) $\langle \alpha, \beta^\vee \rangle = 0 \iff \langle \beta, \alpha^\vee \rangle = 0$.

(ii) $\langle \alpha, \beta^\vee \rangle \langle \beta, \alpha^\vee \rangle = 0, 1, 2, 3$.

If $|\langle \alpha, \beta^\vee \rangle| > 1$ then $|\langle \beta, \alpha^\vee \rangle| = 1$.

(iii) In the four cases of (i) the order of $s_\alpha s_\beta$ is, respectively, $2, 3, 4, 6$.

proof. Denote $V = \mathbb{R} \otimes_{\mathbb{Z}} X$. s_α and s_β map $\mathrm{span}_{\mathbb{R}}(\alpha, \beta) \subseteq V$ to itself. On the basis (α, β) of that space $s_\alpha s_\beta$ is repr. by

$$M_{\alpha\beta} = \begin{pmatrix} \langle \alpha, \beta^\vee \rangle \langle \beta, \alpha^\vee \rangle - 1 & \langle \beta, \alpha^\vee \rangle \\ -\langle \alpha, \beta^\vee \rangle & -1 \end{pmatrix}.$$

As W is finite, $M_{\alpha\beta}^n = I_2$ for some $n \geq 1$.

If $\langle \alpha, \beta^\vee \rangle = 0$, the matrix is triangular and hence satisfies $M_{\alpha\beta}^n = I_n$ only if $\langle \beta, \alpha^\vee \rangle = 0$ too.

Likewise, one proves the converse direction of (i).

The eigenvalues λ_1, λ_2 of $M_{\alpha\beta}$ are two conjugate roots of unity. As $\mathrm{tr}(M_{\alpha\beta}) = \lambda_1 + \lambda_2$, we have

$$|\mathrm{tr}(M_{\alpha\beta})| = |\lambda_1 + \lambda_2| \leq |\lambda_1| + |\lambda_2| = 2.$$

As $\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$ does not have finite index and $M_{\alpha\beta}$ is not the identity matrix, not both eigenvalues can be 1.

Hence $\mathrm{tr}(M_{\alpha\beta}) < 2$ and $\mathrm{tr}(M_{\alpha\beta}) \in \{-2, -1, 0, 1\}$.

This implies (ii).

For (iii) note that any $\varphi \in \mathrm{Aut}(X)$ generated by s_α, s_β is completely determined by its behavior on $\mathrm{span}_{\mathbb{R}}(\alpha, \beta)$:

Choose a W-inv. scalar product on V and decompose

$$V = \mathrm{span}_{\mathbb{R}}(\alpha, \beta) \oplus \mathrm{span}_{\mathbb{R}}(\alpha, \beta)^\perp.$$

As s_α and s_β act on $\mathrm{span}_{\mathbb{R}}(\alpha,\beta)^\perp$ as identity, φ is completely determined by its behavior on $\mathrm{span}_{\mathbb{R}}(\alpha,\beta)$.
In particular, to prove that $s_\alpha s_\beta$ has order n it suffices to show that $M_{\alpha\beta}$ has order n.
The four cases can be calculated "by hand". $\square$