Definition and First Observations

**Def. 1.1** A root datum is a quadruple \( \Phi = (X, R, X^\vee, R^\vee) \) where

- \( X, X^\vee \) free abelian groups of finite rank endowed with a perfect pairing \( \langle \cdot, \cdot \rangle : X \times X^\vee \to \mathbb{Z} \) (i.e., \( \langle \cdot, \cdot \rangle \) identifies each of \( X, X^\vee \) with the dual of the other)
- \( R \subseteq X, R^\vee \subseteq X^\vee \) finite subsets, in bijection via a map \( R \to R^\vee, \alpha \mapsto \alpha^\vee \)

subject to the following conditions:

For \( \alpha \in R \) define endomorphism

\[
S_\alpha : X \to X, \quad S_\alpha : X^\vee \to X^\vee,
\]

\[
S_\alpha(x) = x - \langle x, \alpha^\vee \rangle \alpha \quad \text{and} \quad S_\alpha^\vee(y) = y - \langle \alpha, y \rangle \alpha^\vee
\]

We demand for all \( \alpha \in R \)

- (RD1) \( \langle \alpha, \alpha^\vee \rangle = 2 \),
- (RD2) \( S_\alpha(R) \subseteq R \) and \( S_\alpha^\vee(R^\vee) \subseteq R^\vee \).

**Obs. 1.2** (RD1) implies \( S_\alpha^2 = \text{id}_X \) and \( S_\alpha(\alpha) = -\alpha \)

for \( \alpha \in R \). In particular, \( S_\alpha \in \text{Aut}(X) \) and \( S_\alpha \) acts on \( R \) by permutation.

**Proof.** \( S_\alpha(S_\alpha(\alpha)) = S_\alpha(\alpha) = \langle S_\alpha(\alpha), \alpha^\vee \rangle \alpha \)

\[
= x - \langle x, \alpha^\vee \rangle \alpha - \langle x, \alpha^\vee \rangle \alpha + \langle x, \alpha^\vee \rangle \langle \alpha, \alpha^\vee \rangle = x
\]

\( S_\alpha(\alpha) = x - \langle \alpha, \alpha^\vee \rangle \alpha = -\alpha \).
Observation 1.3: To every root datum \( \Phi = (X, R, X^\vee, R^\vee) \) there is its dual root datum \( \Phi^d = (X^\vee, R^\vee, X, R) \).

Examples 1.4: (i) Endow \( X = X^\vee = \mathbb{Z}^n \) with the standard pairing and define

\[
R = R^\vee = \{ e_i - e_j \mid i \neq j \},
\]

where \( e_i \in \mathbb{Z}^n \) denotes the standard basis vectors.

(ii) Again consider \( X = X^\vee = \mathbb{Z}^n \). Now define

\[
R = \{ \pm 2e_i, \pm e_i \pm e_j \mid i \neq j \},
\]

\[
R^\vee = \{ \pm e_i, \pm e_i \pm e_j \mid i \neq j \}
\]

and a bijection \( \pm 2e_i \mapsto \pm e_i, \pm e_i \pm e_j \mapsto \pm e_i \pm e_j \).

§2 Weyl Group

Def 2.1 Define the Weyl group to be
\[ W = W(\Phi) = \langle s_\alpha | \alpha \in \Phi \rangle \subseteq \text{Aut}(X). \]

Lemma 2.2 The Weyl group \( W \) is finite.

proof. Define the \( \mathbb{R} \)-vector spaces
\[ V = \mathbb{R} \otimes_\mathbb{Z} X, \quad V^\vee = \mathbb{R} \otimes_\mathbb{Z} X^\vee. \]
The perfect pairing \( \langle , \rangle : X \times X^\vee \rightarrow \mathbb{Z} \) extends to a perfect pairing \( \langle , \rangle : V \times V^\vee \rightarrow \mathbb{R} \).
We regard \( \mathbb{R} \) resp. \( R^\vee \) as subsets of \( V \) resp. \( V^\vee \) and the Weyl group as subgroup \( W \subseteq \text{Aut}_\mathbb{R}(V) \) via the embedding
\[ \text{Aut}_2(X) \rightarrow \text{Aut}_\mathbb{R}(V), \quad \psi \mapsto \text{id}_\mathbb{R} \circ \psi. \]
Denote by \( U = \text{span}_\mathbb{R}(R) \subseteq V \) (resp. \( U^\vee = \text{span}_\mathbb{R}(R^\vee) \subseteq V^\vee \)) the subspace of \( V \) (resp. \( V^\vee \)) spanned by \( R \) (resp. \( R^\vee \)).
Consider the \( \mathbb{R} \)-linear map
\[ f: V \rightarrow V^\vee, \quad f(x) = \sum_{\alpha \in \Phi} \langle x, \alpha \rangle \alpha^\vee. \]
Using that \( s_\alpha \) permutes \( R^\vee \), we obtain
\[ f(\alpha) = \frac{1}{2} \langle \alpha, f(\alpha) \rangle \alpha^\vee \quad (\alpha \in \Phi). \]
From
\[ \langle x, f(x) \rangle = \sum_{\alpha \in \Phi} \langle x, \alpha^\vee \rangle \langle x, \alpha \rangle = \sum_{\alpha \in \Phi} \langle x, \alpha \rangle^2 \quad (\ast) \]
it follows \( 0 < \langle \alpha, f(\alpha) \rangle \). Consequently, \( \text{im}(f) = U^\vee = f(U) \) and hence \( V = U + \ker(f) \). Moreover, \((\ast)\) implies
\[ \ker(f) = \{ x \in V \mid \langle x, \alpha \rangle = 0 \text{ for all } \alpha \in \Phi \}. \]
We conclude that $S_\infty$ acts on $\ker(f)$ as the identity and on $U$ by permuting $R$. The assertion transfers to $W = \langle s_\infty | x \in R \rangle$. Thus, $\Phi \in W$ is completely determined by its permutation of $R$. As there are only finitely many permutations, $W$ is finite.

**Def 2.3** Given an Euclidean vector space $(E, (,))$ a **root system** $\Phi$ is a finite set of non-zero vectors satisfying

- $\Phi$ spans $E$.
- For every $\alpha \in \Phi$, the set $\Phi$ is closed under reflections $x \mapsto x - \frac{2(\alpha, x)}{\langle \alpha, \alpha \rangle} \alpha$ through the hyperplane perpendicular to $\alpha$.
- For $\alpha, \beta \in \Phi$, the number $\frac{2(\alpha, \beta)}{\langle \alpha, \alpha \rangle}$ is an integer.

**Obs 2.4** Given a root datum $\Phi = (X, R, X^\vee, R^\vee)$ let $U \subseteq R \otimes_X$ the subspace generated by $R$ (as in the proof of Lemma 1.5). Pick a scalar product $(,)$ on $U$ that is invariant under the Weyl group. (Such an invariant scalar product can be found; since the Weyl group is finite.) Then $R$ is a root system in $U$. The automorphisms on $U$ induced by $S_\infty (\alpha \in R)$ are the reflections in $(U, (,))$. 
§ 3 System of positive roots & reduced root datum

Def 3.1 We call a root datum \( \Phi = (X, R, X^+, R^+) \) reduced if
\[ 2\alpha \not\in R \text{ for any root } \alpha \in R. \]

Def 3.2 Let \( W = W(\Phi) \) be the Weyl group and \( (\cdot, \cdot) \) a \( W \)-inv.
scalar product on \( V = \mathbb{R} \otimes_\mathbb{Q} X \). We call \( R^+ \subseteq R \) a system of
positive roots if there ex. \( x \in V \) s.t. \( (\alpha, x) > 0 \) for all \( \alpha \in R \)
and
\[ R^+ = \{ \alpha \in R \mid (\alpha, x) > 0 \} \]
(or equivalently, if there ex. \( \lambda \in X^* \) s.t. \( \langle \alpha, \lambda \rangle \neq 0 \) for all \( \alpha \in R \)
and
\[ R^+ = \{ \alpha \in R \mid \langle \alpha, \lambda \rangle > 0 \} \].

Obs 3.3 (1) The convex hull of \( R^+ \) in \( V \) does not contain \( 0 \).
(2) \( R \) is the disjoint union of \( R^+ \) and \( -R^+ \).
(3) If \( \alpha, \beta \in R^+ \) s.t. \( \alpha + \beta \in R \) then \( \alpha + \beta \in R^+ \).
(4) \((R^+)^\vee \) is a system of positive roots in \( R^v \),
i.e., there is an \( x \in X \) s.t. \( \langle x, \alpha \rangle \neq 0 \)
for all \( \alpha \in R \) and
\[ (R^+)^\vee = \{ \alpha \in R^v \mid \langle x, \alpha \rangle > 0 \} \].
§ 4 Two roots

Lemma 4.1 Let $\alpha, \beta \in \mathbb{R}$ be lin. indep. roots

(i) $\langle \alpha, \beta^* \rangle = 0 \iff \langle \beta, \alpha^* \rangle = 0$.

(ii) $\langle \alpha, \beta^* \rangle < \langle \beta, \alpha^* \rangle = 0, 1, 2, 3$.

If $|\langle \alpha, \beta^* \rangle | > 1$ then $|\langle \beta, \alpha^* \rangle | = 1$.

(iii) In the four cases of (i) the order of $s_\alpha s_\beta$ is, respectively, 2, 3, 4, 6.

proof. Denote $V = \mathbb{R}^2 \times \mathbb{R}$. $s_\alpha$ and $s_\beta$ map $\text{span}_\mathbb{R}(\alpha, \beta) \subseteq V$ to itself. On the basis $(\alpha, \beta)$ of that space $s_\alpha s_\beta$ is rep. by

$$M_{\alpha \beta} = \begin{pmatrix} \langle \alpha, \beta^* \rangle & \langle \beta, \alpha^* \rangle - i \langle \beta, \alpha^* \rangle \\ - i \langle \alpha, \beta^* \rangle & 1 \end{pmatrix}.$$ 

As $W$ is finite, $M_{\alpha \beta}^n = I_2$ for some $n \geq 1$.

If $\langle \alpha, \beta^* \rangle = 0$, the matrix is triangular and hence satisfies $M_{\alpha \beta}^n = I_2$ only if $\langle \beta, \alpha^* \rangle = 0$ too.

Likewise, one proves the converse direction of (i).

The eigenvalues $\lambda_1, \lambda_2$ of $M_{\alpha \beta}$ are two conjugate roots of unity. As $\text{tr}(M_{\alpha \beta}) = \lambda_1 + \lambda_2$, we have

$$|\text{tr}(M_{\alpha \beta})| = |\lambda_1 + \lambda_2| \leq |\lambda_1| + |\lambda_2| = 2.$$ 

As (i) does not have finite index and $M_{\alpha \beta}$ is not the identity matrix, not both eigenvalues can be 1.

Hence $\text{tr}(M_{\alpha \beta}) < 2$ and $\text{tr}(M_{\alpha \beta}) \in \{-2, -1, 0, 1\}$.

This implies (ii).

For (iii) note that any $c \in \text{Aut}(V)$ generated by $s_\alpha, s_\beta$ is completely determined by its behavior on $\text{span}_\mathbb{R}(\alpha, \beta)$.

Choose a $W$-inv. scalar product on $V$ and decompose $V = \text{span}_\mathbb{R}(\alpha, \beta) \oplus \text{span}_\mathbb{R}(\alpha, \beta)^*$. 
As \( \sigma \) and \( \tau \) act on \( \text{span}_\mathbb{R}(\alpha, \beta)^\perp \) as identity, \( \Phi \) is completely determined by its behavior on \( \text{span}_\mathbb{R}(\alpha, \beta) \).

In particular, to prove that \( \Sigma \Phi \) has order \( n \) it suffices to show that \( M \alpha \beta \) has order \( n \).

The four cases can be calculated "by hand". \( \square \)