Finite Type, Separated and Proper Morphisms

TECHNISCHE UNIVERSITÄT DARMSTADT

by Christopher Lang

Finite Type over k

Idea: a scheme with finite k-dimension.

Finite Type over *k*

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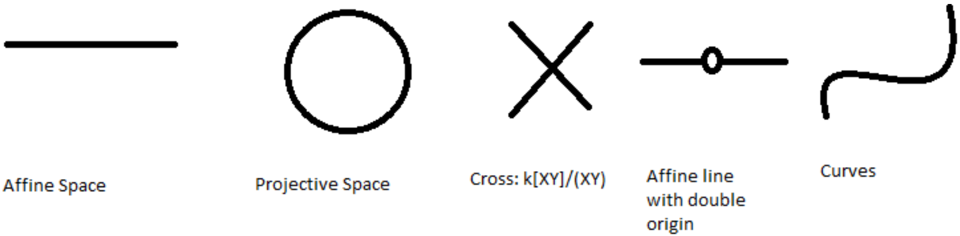

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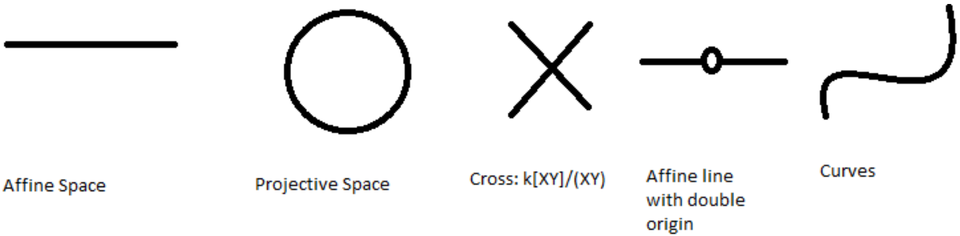

In general: locally of the form $\mathrm{Spec}(k[X_1, \ldots, X_n]/\mathrm{ideal})$

Morphisms of finite type

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f is of finite type, if

it is locally of finite type and quasi-compact (i.e. preimages of q.c. opens are q.c.)

Properties of morphisms of finite type

Local on the target

It suffices that for *some* open cover $Y = \bigcup_i \operatorname{Spec}(A_i)$ and $f^{-1}(\operatorname{Spec}(A_i)) = \bigcup_j \operatorname{Spec}(B_{ij})$, the morphisms $f|_{\operatorname{Spec}(B_{ij})} : \operatorname{Spec}(B_{ij}) \to \operatorname{Spec}(A_i)$ come from finitely generated algebras $A_i \to B_{ij}$

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Stable under composition

Proof: Reduce to affine schemes. Now, if $A \to B$ and $B \to C$ are fin. gen. algebras, $A \to C$ is as well

Properties of morphisms of finite type

Stable under base change

$$\begin{array}{ccc} X \times_Y Z & \longrightarrow & X \\ {\scriptstyle \text{of finite type}}\downarrow & & \downarrow{\scriptstyle f} \\ Z & \longrightarrow & Y \end{array}$$

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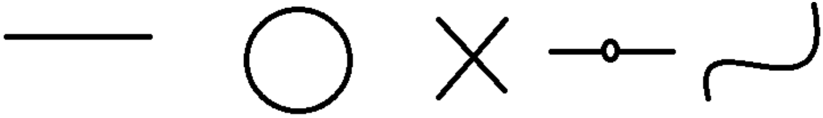
Proofidea: Reduce to affine schemes. If $A \to B$ is a fin. gen. algebra, then $C \to B \otimes_A C$ is as well for any A-algebra C (with generators $b_i \otimes 1_C$)

„Hausdorff“ schemes

Question: Which schemes should be „hausdorff“ and which not?



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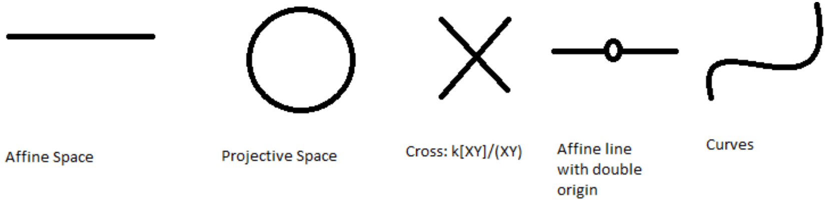
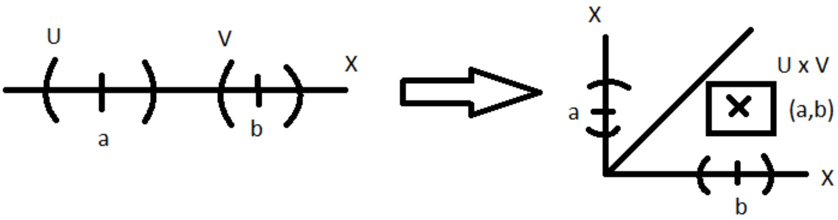


Problem: Schemes are almost never hausdorff in the usual sense, as there are generic points etc.
$\rightsquigarrow$We need a different definition for a scheme to be „hausdorff“.

Hausdorff spaces
topological



X is Hausdorff
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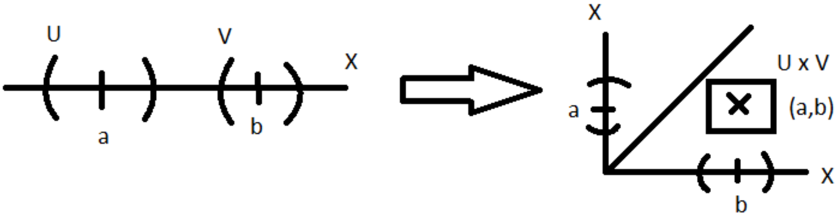

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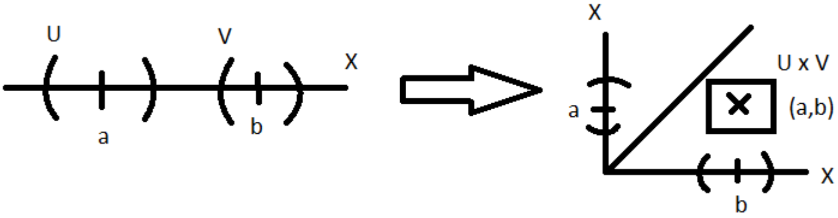

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$\Leftrightarrow$ For every space Y and map $f : Y \to X$, the graph $\Gamma_f \subseteq Y \times X$ is closed
$\Leftrightarrow$ For all maps $f, g : Y \to X$, the equalizer $\{y \in Y : f(y) = g(y)\} \subseteq Y$ is closed

Diagonal morphism

Let $X \to S$ be a scheme over S.

$$X \xrightarrow{\Delta_X} X \times_S X$$

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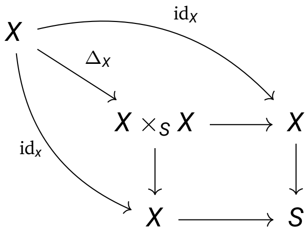
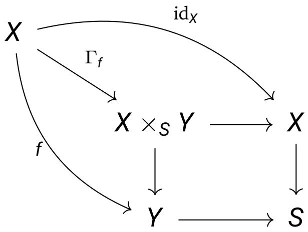


$\Delta_X : X \to X \times_S X$ is called the *diagonal* morphism of X (over S)

Graph as a morphism

Let $f : X \to Y$ be a morphism of S-schemes.



$\Gamma_f : X \to X \times_S Y$ is called the *graph* (morphism) of f

Equalizer scheme

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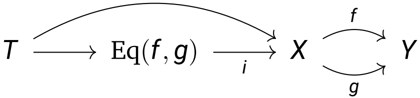

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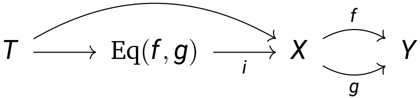

The *equalizer* of f and g is a scheme $\mathrm{Eq}(f, g)$ over S together with a morphism $i : \mathrm{Eq}(f, g) \to X$ with

$$\mathrm{Eq}(f, g)_S(T) \xrightarrow{\sim} \{x \in X_S(T) : f \circ x = g \circ x \quad (\text{in } Y_S(T))\}$$

Separated morphisms

The right way to define hausdorff for schemes

A morphism $Y \to S$ is *separated*, if

the three equivalent conditions are true:

1. $\Delta_Y : Y \to Y \times_S Y$ is a closed immersion
2. For all $f : X \to Y$, the graph $\Gamma_f : X \to X \times_S Y$ is a closed immersion
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A S-scheme Y is separated, if $Y \to S$ is a separated morphism.

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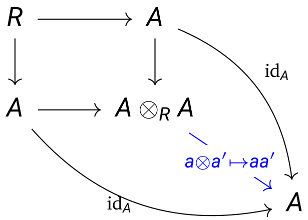

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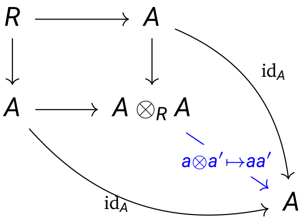

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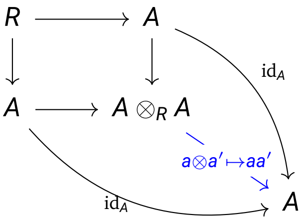

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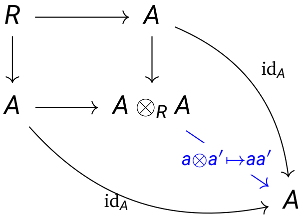

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A bit more complicated, but true: $\mathbb{P}^n_R$ is separated (over the Ring R)

A counterexample

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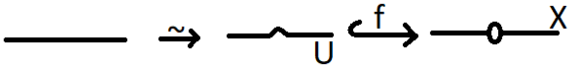

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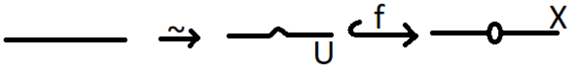

Then $\mathrm{Eq}(f, g) = U \cap V \cong D(0) \subset \mathbb{A}^1_k$, which is *not* a closed immersion!

Properties of separated morphisms

Let $X \to S$ be separated. (Hence $\Delta : X \to X \times_S X$ is a closed immersion.)

Stable under base change

If $S \to S'$ is a base change, then $\Delta' : X \times_S S' \to (X \times_S S') \times_{S'} (X \times_S S') \cong (X \times_S X) \times_S S'$ is just the base change of Δ

(Remember: $S' \times_{S'} X \cong X$)

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Local on the target

$X \times_S X$ can be computed locally in S, and closed immersions (here: of Δ) are local on the target

Properties of separated morphisms

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Let $f : X \to Y$ and $g : Y \to Z$ be separated.

$$
\begin{array}{ccccc}
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„Compact“ schemes

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No:
Affine space $\mathbb{A}_k^n$ (since $\mathbb{C}^n$ is not compact)

Yes:
Projective space $\mathbb{P}_k^n$ (as $\mathbb{S}^n$ is compact)
Closed subspaces of compact spaces

Proper maps

In topology:

The map $f : X \to Y$ is proper, if

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In good circumstances (locally compact and hausdorff):
f is proper $\Leftrightarrow$ f is universally closed (i.e. closed after any base change)

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Recall: $f : X \to Y$ is universally closed, if for all schemes Z the base change $f_Z : X \times_Y Z \to Z$ is closed (as a map of topological spaces).

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Proper morphisms are

stable under composition, base change and local on the target.

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