

# Finite Type, Separated and Proper Morphisms



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# Finite Type over $k$

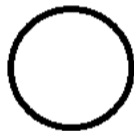
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Affine Space



Projective Space



Cross:  $k[XY]/(XY)$



Affine line  
with double  
origin



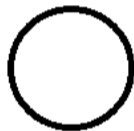
Curves

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In general: locally of the form  $\text{Spec}(k[X_1, \dots, X_n]/\text{ideal})$

# Morphisms of finite type

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$f : X \rightarrow Y$  is locally of finite type, if

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$f$  is of finite type, if

it is locally of finite type and quasi-compact (i.e. preimages of q.c. opens are q.c.)

# Properties of morphisms of finite type

## Local on the target

It suffices that for *some* open cover  $Y = \bigcup_i \text{Spec}(A_i)$  and  $f^{-1}(\text{Spec}(A_i)) = \bigcup_j \text{Spec}(B_{ij})$ , the morphisms  $f|_{\text{Spec}(B_{ij})} : \text{Spec}(B_{ij}) \rightarrow \text{Spec}(A_i)$  come from finitely generated algebras  $A_i \rightarrow B_{ij}$



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## Stable under composition

Proof: Reduce to affine schemes. Now, if  $A \rightarrow B$  and  $B \rightarrow C$  are fin. gen. algebras,  $A \rightarrow C$  is as well

# Properties of morphisms of finite type

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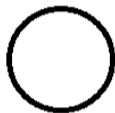
Proof idea: Reduce to affine schemes. If  $A \rightarrow B$  is a fin. gen. algebra, then  $C \rightarrow B \otimes_A C$  is as well for any  $A$ -algebra  $C$  (with generators  $b_i \otimes 1_C$ )

# „Hausdorff“ schemes

Question: Which schemes should be „hausdorff“ and which not?



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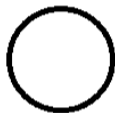
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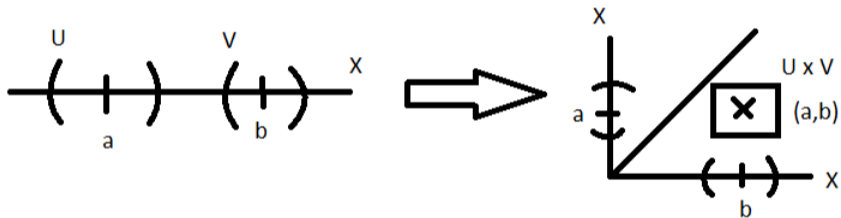


Curves

Problem: Schemes are almost never hausdorff in the usual sense, as there are generic points etc.  
↪ We need a different definition for a scheme to be „hausdorff“.

# Hausdorff spaces

topological

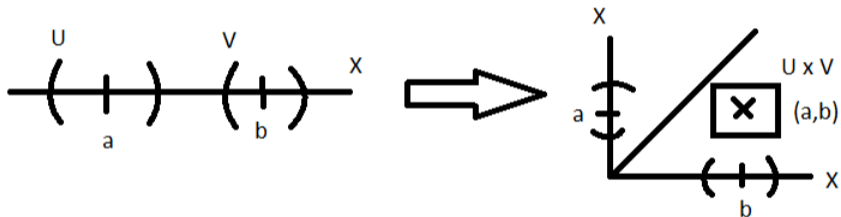


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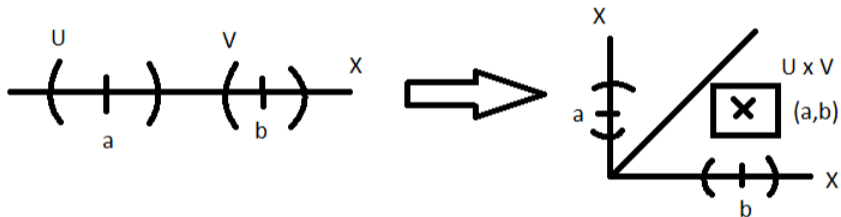
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$\Leftrightarrow$  For all maps  $f, g : Y \rightarrow X$ , the equalizer  $\{y \in Y : f(y) = g(y)\} \subseteq Y$  is closed



# Diagonal morphism

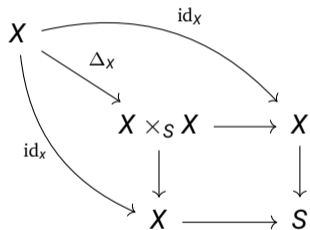
Let  $X \rightarrow S$  be a scheme over  $S$ .

$$\begin{array}{ccc} X & & \\ & \searrow^{\Delta_X} & \\ & & X \times_S X \end{array}$$

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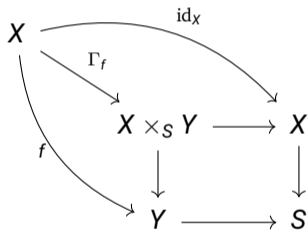
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# Graph as a morphism

Let  $f : X \rightarrow Y$  be a morphism of  $S$ -schemes.



$\Gamma_f : X \rightarrow X \times_S Y$  is called the *graph* (morphism) of  $f$

## Equalizer scheme

Let  $f, g : X \rightarrow Y$  be two  $S$ -morphisms.

Idea:  $\text{Eq}(f, g) = \{x \in X : f(x) = g(x)\}$ . This is not a scheme...

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Solution: Use  $T$ -valued points! (For some  $S$ -scheme  $T$ )

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The *equalizer* of  $f$  and  $g$  is a scheme  $\text{Eq}(f, g)$  over  $S$  together with a morphism  $i : \text{Eq}(f, g) \rightarrow X$  with

$$\text{Eq}(f, g)_S(T) \xrightarrow{\sim} \{x \in X_S(T) : f \circ x = g \circ x \quad (\text{in } Y_S(T))\}$$

# Separated morphisms

The right way to define hausdorff for schemes

A morphism  $Y \rightarrow S$  is *separated*, if

the three equivalent conditions are true:

1.  $\Delta_Y : Y \rightarrow Y \times_S Y$  is a closed immersion
2. For all  $f : X \rightarrow Y$ , the graph  $\Gamma_f : X \rightarrow X \times_S Y$  is a closed immersion
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A  $S$ -scheme  $Y$  is separated, if  $Y \rightarrow S$  is a separated morphism.



## Examples of separated schemes

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$$\begin{array}{ccc} R & \longrightarrow & A \\ \downarrow & & \downarrow \\ A & \longrightarrow & A \otimes_R A \end{array} \begin{array}{c} \xrightarrow{\text{id}_A} \\ \downarrow \\ \xrightarrow{\text{id}_A} \end{array} \begin{array}{c} A \\ \\ A \end{array}$$

$a \otimes a' \mapsto aa'$

The diagram illustrates the relationship between the ring  $R$ , the ring  $A$ , and the tensor product  $A \otimes_R A$ . It consists of a commutative square with two additional maps. The top row is  $R \rightarrow A$ . The bottom row is  $A \rightarrow A \otimes_R A$ . A vertical arrow points from  $R$  to  $A$ , and another from  $A$  to  $A \otimes_R A$ . A curved arrow labeled  $\text{id}_A$  goes from  $A$  (top right) to  $A$  (bottom right). Another curved arrow labeled  $\text{id}_A$  goes from  $A$  (bottom left) to  $A$  (bottom right). A blue arrow labeled  $a \otimes a' \mapsto aa'$  points from  $A \otimes_R A$  to  $A$  (bottom right).

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In particular:  $\mathbb{A}_R^n$  is separated (over the Ring  $R$ )



## A counterexample

The affine line with double origin

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Let  $f : \mathbb{A}_k^1 \rightarrow U \hookrightarrow X$  and  $g : \mathbb{A}_k^1 \rightarrow V \hookrightarrow X$  be the „usual“ morphisms

The diagram illustrates the relationship between the affine line with double origin and the affine line. It consists of three horizontal lines. The leftmost line is a simple horizontal line. An arrow labeled with a tilde (~) points from this line to the middle line. The middle line is a horizontal line with a wavy break in the center, and the letter 'U' is written below the break. An arrow labeled 'f' points from the middle line to the rightmost line. The rightmost line is a horizontal line with a small circle (representing an origin) on it, and the letter 'X' is written at the far right end.

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Then  $\text{Eq}(f, g) = U \cap V \cong D(0) \subset \mathbb{A}_k^1$ , which is *not* a closed immersion!



## Properties of separated morphisms

Let  $X \rightarrow S$  be separated. (Hence  $\Delta : X \rightarrow X \times_S X$  is a closed immersion.)

### Stable under base change

If  $S \rightarrow S'$  is a base change, then  $\Delta' : X \times_S S' \rightarrow (X \times_S S') \times_{S'} (X \times_S S') \cong (X \times_S X) \times_S S'$  is just the base change of  $\Delta$

(Remember:  $S' \times_{S'} X \cong X$ )

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### Local on the target

$X \times_S X$  can be computed locally in  $S$ , and closed immersions (here: of  $\Delta$ ) are local on the target

# Properties of separated morphisms

## Stable under composition

Let  $f : X \rightarrow Y$  and  $g : Y \rightarrow Z$  be separated.

$$\begin{array}{ccccc} X & \xrightarrow{\Delta_f} & X \times_Y X & \longrightarrow & Y \\ & \searrow \Delta_{gof} & \downarrow & \square & \downarrow \Delta_g \\ & & X \times_Z X & \longrightarrow & Y \times_Z Y \end{array}$$

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# „Compact“ schemes

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No:

Affine space  $\mathbb{A}_k^n$  (since  $\mathbb{C}^n$  is not compact)

Yes:

Projective space  $\mathbb{P}_k^n$  (as  $\mathbb{S}^n$  is compact)

Closed subspaces of compact spaces

# Proper maps

In topology:

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In good circumstances (locally compact and hausdorff):

$f$  is proper  $\Leftrightarrow f$  is universally closed (i.e. closed after any base change)

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Recall:  $f : X \rightarrow Y$  is universally closed, if for all schemes  $Z$  the base change  $f_Z : X \times_Y Z \rightarrow Z$  is closed (as a map of topological spaces).

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Proper morphisms are

stable under composition, base change and local on the target.



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