Finite Type, Separated and Proper Morphisms

by Christopher Lang
Finite Type over $k$

Idea: a scheme with finite $k$-dimension.
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- Affine Space
- Projective Space
- Cross: $k[XY]/(XY)$
- Affine line with double origin
- Curves
Finite Type over $k$

Idea: a scheme with finite $k$-dimension.

In general: locally of the form $\text{Spec}(k[X_1, \ldots, X_n]/\text{ideal})$
Morphisms of finite type

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$f : X \to Y$ is locally of finite type, if

For any $\text{Spec}(A) \subseteq Y$ and for any $\text{Spec}(B) \subseteq f^{-1}(\text{Spec}(A))$, the algebra $A \to B$ is finitely generated
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$f$ is of finite type, if

it is locally of finite type and quasi-compact (i.e. preimages of q.c. opens are q.c.)
Properties of morphisms of finite type

Local on the target

It suffices that for some open cover $Y = \bigcup_i \text{Spec}(A_i)$ and $f^{-1}(\text{Spec}(A_i)) = \bigcup_j \text{Spec}(B_{ij})$, the morphisms $f|_{\text{Spec}(B_{ij})} : \text{Spec}(B_{ij}) \rightarrow \text{Spec}(A_i)$ come from finitely generated algebras $A_i \rightarrow B_{ij}$. 

Stable under composition

Proof: Reduce to affine schemes. Now, if $A \rightarrow B$ and $B \rightarrow C$ are finitely generated algebras, then $A \rightarrow C$ is as well.
Properties of morphisms of finite type

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Proof: Reduce to affine schemes. Now, if \( A \to B \) and \( B \to C \) are fin. gen. algebras, \( A \to C \) is as well.
Properties of morphisms of finite type

Stable under base change

\[ X \times_Y Z \rightarrow X \]
\[ \quad \text{of finite type} \]
\[ \quad \downarrow f \]
\[ Z \rightarrow Y \]
Properties of morphisms of finite type

Stable under base change

Proof idea: Reduce to affine schemes. If $A \rightarrow B$ is a fin. gen. algebra, then $C \rightarrow B \otimes_A C$ is as well for any $A$-algebra $C$ (with generators $b_i \otimes 1_C$)
Question: Which schemes should be “haussdorff” and which not?

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- Projective Space
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„Hausdorff“ schemes

Question: Which schemes should be „hausdorff“ and which not?

- Affine Space
- Projective Space
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Problem: Schemes are almost never hausdorff in the usual sense, as there are generic points etc. We need a different definition for a scheme to be „hausdorff“.
$X$ is Hausdorff
$\iff$ The diagonal $\Delta_X \subseteq X \times X$ is closed
Hausdorff spaces

topological

\[ X \text{ is Hausdorff} \iff \text{The diagonal } \Delta_X \subseteq X \times X \text{ is closed} \]
\[ \iff \text{For every space } Y \text{ and map } f : Y \to X, \text{ the graph } \Gamma_f \subseteq Y \times X \text{ is closed} \]
Hausdorff spaces

topological

$X$ is Hausdorff
$\iff$ The diagonal $\Delta_X \subseteq X \times X$ is closed
$\iff$ For every space $Y$ and map $f : Y \to X$, the graph $\Gamma_f \subseteq Y \times X$ is closed
$\iff$ For all maps $f, g : Y \to X$, the equalizer $\{ y \in Y : f(y) = g(y) \} \subseteq Y$ is closed
Let $X \rightarrow S$ be a scheme over $S$.

$\Delta_X : X \rightarrow X \times_S X$ is called the \textit{diagonal} morphism of $X$ (over $S$)
Let $X \to S$ be a scheme over $S$.

$\Delta_X : X \to X \times_S X$ is called the *diagonal* morphism of $X$ (over $S$)
Graph as a morphism

Let $f : X \to Y$ be a morphism of $S$-schemes.

\[
\begin{array}{c}
X \\
\downarrow f \\
X \times_S Y \\
\downarrow \Gamma_f \\
Y \\
\downarrow \Gamma_f \\
S
\end{array}
\]

$\Gamma_f : X \to X \times_S Y$ is called the graph (morphism) of $f$
Let $f, g : X \to Y$ be two $S$-morphisms.

Idea: $\text{Eq}(f, g) = \{x \in X : f(x) = g(x)\}$. This is not a scheme...

Solution: Use $T$-valued points! (For some $S$-scheme $T$)

The equalizer of $f$ and $g$ is a scheme $\text{Eq}(f, g)$ over $S$ together with a morphism $i : \text{Eq}(f, g) \to X$ with $\text{Eq}(f, g) \times_S T \to \{x \in X_S(T) : f(x) = g(x)\}$. (in $Y_S(T)$)
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$$
\begin{array}{cc}
T & \xrightarrow{i} \text{Eq}(f, g) \\
\text{Eq}(f, g) & \xleftarrow{i} X & \xrightarrow{f} Y \\
X & \xleftarrow{g} Y
\end{array}
$$
Equalizer scheme

Let $f, g : X \rightarrow Y$ be two $S$-morphisms.

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Solution: Use $T$-valued points! (For some $S$-scheme $T$)

$$
\begin{array}{ccc}
T & \longrightarrow & \text{Eq}(f, g) \\
& \downarrow i & \nearrow f \\
& & X \\
& \downarrow g & \searrow Y
\end{array}
$$

The equalizer of $f$ and $g$ is a scheme $\text{Eq}(f, g)$ over $S$ together with a morphism $i : \text{Eq}(f, g) \rightarrow X$ with

$$
\text{Eq}(f, g)_S(T) \xrightarrow{i} \{x \in X_S(T) : f \circ x = g \circ x \text{ (in } Y_S(T))\}
$$
### Separated morphisms
The right way to define hemdorff for schemes

A morphism $Y \to S$ is *separated*, if

the three equivalent conditions are true:

1. $\Delta_Y : Y \to Y \times_S Y$ is a closed immersion
2. For all $f : X \to Y$, the graph $\Gamma_f : X \to X \times_S Y$ is a closed immersion
3. For all $f, g : X \to Y$, the equalizer $i : \text{Eq}(f, g) \to X$ is a closed immersion
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A $S$-scheme $Y$ is separated, if $Y \to S$ is a separated morphism.
Affine schemes: Let $\text{Spec}(A) \to \text{Spec}(R)$.

The map $a \otimes a' \mapsto aa'$ is surjective $\Rightarrow \Delta \text{Spec}(A)$ is a closed immersion $\Rightarrow \text{Spec}(A)$ is separated (over $R$).

In particular: $A^n_R$ is separated (over the ring $R$).

A bit more complicated, but true: $P^n_R$ is separated (over the ring $R$).
Examples of separated schemes

Affine schemes: Let $\text{Spec}(A) \to \text{Spec}(R)$. Then $\Delta_{\text{Spec}(A)}$ comes from

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$$\begin{array}{ccc}
R & \rightarrow & A \\
\downarrow & & \downarrow \\
A & \rightarrow & A \otimes_R A
\end{array}$$

The map $a \otimes a' \mapsto aa'$ is surjective $\Rightarrow \Delta_{\text{Spec}(A)}$ is a closed immersion $\Rightarrow \text{Spec}(A)$ is separated (over $R$)
Affine schemes: Let $\text{Spec}(A) \to \text{Spec}(R)$. Then $\Delta_{\text{Spec}(A)}$ comes from

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In particular: $\mathbb{A}^n_R$ is separated (over the Ring $R$)
A bit more complicated, but true: $\mathbb{P}^n_R$ is separated (over the Ring $R$)
Let $X$ be the affine line (over $k$) with double origin, covered by $U \cong V \cong \mathbb{A}^1_k$. The affine line with double origin
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We check the equalizer condition:

Let $f : \mathbb{A}^1_k \rightarrow U \hookrightarrow X$ and $g : \mathbb{A}^1_k \rightarrow V \hookrightarrow X$ be the „usual“ morphisms.

\[
\begin{array}{cccc}
& & & X \\
& & \sim & \hookrightarrow \ & U \\
\end{array}
\]
A counterexample
The affine line with double origin

Let $X$ be the affine line (over $k$) with double origin, covered by $U \cong V \cong \mathbb{A}^1_k$

We check the equalizer condition:

Let $f : \mathbb{A}^1_k \to U \hookrightarrow X$ and $g : \mathbb{A}^1_k \to V \hookrightarrow X$ be the “usual” morphisms

Then $\text{Eq}(f, g) = U \cap V \cong D(0) \subset \mathbb{A}^1_k$, which is not a closed immersion!
Properties of separated morphisms

Let $X \to S$ be separated. (Hence $\Delta : X \to X \times_S X$ is a closed immersion.)

Stable under base change

If $S \to S'$ is a base change, then $\Delta' : X \times_S S' \to (X \times_S S') \times_{S'} (X \times_S S') \cong (X \times_S X) \times_S S'$ is just the base change of $\Delta$

(Remember: $S' \times_{S'} X \cong X$)
*Properties of separated morphisms*

Let $X \to S$ be separated. (Hence $\Delta : X \to X \times_S X$ is a closed immersion.)

**Stable under base change**

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(Remember: $S' \times_{S'} X \cong X$)

**Local on the target**

$X \times_S X$ can be computed locally in $S$, and closed immersions (here: of $\Delta$) are local on the target.
Properties of separated morphisms

Stable under composition

Let \( f : X \rightarrow Y \) and \( g : Y \rightarrow Z \) be separated.

\[
\begin{array}{cccccc}
X & \xrightarrow{\Delta_f} & X \times_YY & \xrightarrow{\Delta_g} & Y \\
\downarrow^{\Delta_{gof}} & & \downarrow^{\Box} & & \downarrow^{\Delta_g} \\
X \times_YY & \xrightarrow{} & Y \times_YY
\end{array}
\]

(Closed immersions are red!)
Properties of separated morphisms

Stable under composition

Let $f : X \to Y$ and $g : Y \to Z$ be separated.

(Closed immersions are red!)
Properties of separated morphisms

Stable under composition

Let $f : X \to Y$ and $g : Y \to Z$ be separated.

\[ X \xrightarrow{\Delta_f} X \times_Y X \xrightarrow{\square} Y \]

\[ X \times_Z X \xrightarrow{\Delta_{gof}} Y \times_Z Y \]

(Closed immersions are red!)
Which schemes should be „compact“?

No:
- Affine space $\mathbb{A}^n_k$ (since $\mathbb{C}^n$ is not compact)

Yes:
- Projective space $\mathbb{P}^n_k$ (as $\mathbb{S}^n$ is compact)
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Yes:
Projective space $\mathbb{P}^n_k$ (as $S^n$ is compact)
Closed subspaces of compact spaces
Proper maps

In topology:

The map \( f : X \to Y \) is proper, if

Preimages of compact sets are compact
Proper maps

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Hence: \( X \) is compact \( \iff \) \( X \to \{ * \} \) is proper
Proper maps

In topology:

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Preimages of compact sets are compact

Hence: $X$ is compact $\iff X \to \{\ast\}$ is proper

Problem for schemes: Too many morphisms are “proper” in the usual sense (namely all quasi-compact morphisms)
Proper maps

In topology:

The map \( f : X \rightarrow Y \) is proper, if

Preimages of compact sets are compact

Hence: \( X \) is compact \( \iff \) \( X \rightarrow \{\ast\} \) is proper

Problem for schemes: Too many morphisms are „proper“ in the usual sense (namely all quasi-compact morphisms)

In good circumstances (locally compact and hausdorff):
\( f \) is proper \( \iff \) \( f \) is universally closed (i.e. closed after any base change)
A morphism of schemes is proper, if it is

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1. of finite type,
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Recall: $f : X \to Y$ is universally closed, if for all schemes $Z$ the base change $f_Z : X \times_Y Z \to Z$ is closed (as a map of topological spaces).
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1. of finite type,
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Recall: $f : X \to Y$ is universally closed, if for all schemes $Z$ the base change $f_Z : X \times_Y Z \to Z$ is closed (as a map of topological spaces).

Proper morphisms are

stable under composition, base change and local on the target.
Examples of proper morphisms

Projective space $\mathbb{P}^n_k \to \text{Spec}(k)$ and closed immersions are proper.
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Affine space $\mathbb{A}^1_k \to \text{Spec}(k)$ is *not* universally closed (and hence not proper):
Examples of proper morphisms

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Base change by $\mathbb{A}^1_k$:
Examples of proper morphisms

Projective space $\mathbb{P}^n_k \to \text{Spec}(k)$ and closed immersions are proper.

Affine space $\mathbb{A}^1_k \to \text{Spec}(k)$ is not universally closed (and hence not proper):

Base change by $\mathbb{A}^1_k$: 

$$
\begin{align*}
\mathbb{A}^2_k & \longrightarrow \mathbb{A}^1_k \\
\downarrow & \quad \downarrow \\
\mathbb{A}^1_k & \longrightarrow \text{Spec}(k)
\end{align*}
$$