$$Sp(V, \langle\,,\rangle)$$

$\langle\,,\rangle : V \times V \longrightarrow \mathbb{R}$ nicht ausgeartet, alternierend

$$\langle v, v\rangle = 0$$

$$0 = \langle v+w, v+w\rangle = \langle v,w\rangle + \langle w,v\rangle = 0$$

$$\langle v,w\rangle = {}^t v J w \qquad J = \begin{pmatrix} & & & & & 1 \\ & & & & \iddots & \\ & & & 1 & & \\ & & -1 & & & \\ & \iddots & & & & \\ -1 & & & & & \end{pmatrix}$$

$$= \begin{pmatrix} & K \\ -K & \end{pmatrix} \in GL_{2n}$$

$$K = \begin{pmatrix} & & 1 \\ & \iddots & \\ 1 & & \end{pmatrix}$$

$$Sp_{2n} = \{ X \in GL_{2n} \mid {}^t X J X = J \}$$

$$= \left\{ \begin{pmatrix} A & B \\ C & D \end{pmatrix} \;\middle|\; {}^tCKA = {}^tAKC,\ {}^tDKB = {}^tBKD,\ {}^tAKD - {}^tCKB = K \right\}$$

$$X = \begin{array}{c} \begin{pmatrix} A & B \\ C & D \end{pmatrix} \end{array} \begin{array}{l} \}\, n \\ \}\, n \end{array}$$

$$\begin{pmatrix} {}^tA & {}^tC \\ {}^tB & {}^tD \end{pmatrix}\begin{pmatrix} 0 & K \\ -K & 0 \end{pmatrix}\begin{pmatrix} A & B \\ C & D \end{pmatrix} = \begin{pmatrix} 0 & K \\ -K & 0 \end{pmatrix}$$

$$\underbrace{\begin{pmatrix} {}^tA & {}^tC \\ {}^tB & {}^tD \end{pmatrix}\begin{pmatrix} 0 & K \\ -K & 0 \end{pmatrix}}_{\begin{pmatrix} -{}^tCK & {}^tAK \\ -{}^tDK & {}^tBK \end{pmatrix}}$$

$$\begin{pmatrix} -{}^tCKA + {}^tAKC & -{}^tCKB + {}^tAKD \\ -{}^tDKA + {}^tBKC & -{}^tDKB + {}^tBKD \end{pmatrix}$$

$$T = \left\{ \begin{pmatrix} * & & \\ & \ddots & \\ & & * \end{pmatrix} \in GL_{2n} \right\} \cap Sp_{2n} = \left\{ \begin{pmatrix} t_1 & & & & \\ & \ddots & & & \\ & & t_n & & \\ & & & t_n^{-1} & \\ & & & & \ddots \\ & & & & & t_n^{-1} \end{pmatrix} \right\}$$

${}^tAKD = K$, A, D Diag.

$A = \begin{pmatrix} t_1 & & \\ & \ddots & \\ & & t_n \end{pmatrix}$, $D = \begin{pmatrix} t_{n+1} & & \\ & \ddots & \\ & & t_{2n} \end{pmatrix}$

$$KD = \begin{pmatrix} & & t_{2n} \\ & \iddots & \\ t_{n+1} & & \end{pmatrix}$$

$$T \cong \mathbb{G}_m^n \implies X^*(T) \cong \mathbb{Z}^n$$

$$X^*(T) = \{(d_1, \ldots, d_{2n}) \in \mathbb{Z}^{2n} \mid d_1 + d_{2n} = \cdots = d_n + d_{n+1} = 0\}$$

$$\mathrm{Lie}(Sp_{2n}) = \{X \in M_{2n}(k) \mid {}^tX J + J X = 0\}$$

$$= \left\{ \begin{pmatrix} A & B \\ C & D \end{pmatrix} \;\middle|\; \begin{array}{l} KC = {}^tCK \\ KB = {}^tBK \end{array} \quad KD + {}^tAK = 0 \right\}$$

$$\begin{aligned} Ad: T &\longrightarrow GL(\mathrm{Lie}(Sp_{2n})) \\ t &\longmapsto \left(\begin{pmatrix} A & B \\ C & D \end{pmatrix} \longmapsto t^{-1} \begin{pmatrix} A & B \\ C & D \end{pmatrix} t \right) \end{aligned}$$

$$\implies \ldots \implies$$

$$X^*(T) = \mathbb{Z}^n$$

$$\left(\begin{pmatrix} t_1 & & & & & \\ & \ddots & & & & \\ & & t_n & & & \\ & & & t_n^{-1} & & \\ & & & & \ddots & \\ & & & & & t_1^{-1} \end{pmatrix} \mapsto \underset{\updownarrow}{t_1^{d_1} \cdots t_n^{d_n}} \right) \longleftrightarrow (d_1, \ldots, d_n)$$

$$R = \{\pm e_i \pm e_j \mid 1 \le i < j \le n\}$$
$$\cup \{2e_i \mid 1 \le i \le n\}$$

$\subseteq \mathbb{Z}^n$

$X_*(T) = \mathbb{Z}^n$, $\underbrace{\mathbb{Z}^n}_{X^*(T)} \times \underbrace{\mathbb{Z}^n}_{X_*(T)} \longrightarrow \mathbb{Z}$ standard pairing

$R^\vee$: $(\pm e_i \pm e_j)^\vee = \pm e_i \pm e_j$

$(2e_i)^\vee = e_i$

$R^\vee = \{\pm e_i \pm e_j \mid 1 \leq i < j \leq n\}$

$\cup \{e_i \mid 1 \leq i \leq n\}$

Wurzeldatum: $(\mathbb{Z}^n, R, \mathbb{Z}^n, R^\vee)$

$\mathrm{Cent}(Sp_{2n}) = \mu_2 \iff \mathbb{Z}^n / \langle R \rangle_{\mathbb{Z}} \cong \mathbb{Z}/2\mathbb{Z}$

$\mathbb{Z}^n / \langle R^\vee \rangle_{\mathbb{Z}} = 1 = \pi_1(Sp_{2n})$

$\iff$: Sp_{2n} einfach zusammenhängend

$\pi_1(GL_n) = \mathbb{Z} = \pi_1(GL_n(\mathbb{C}))$

$$\pi_1^{\acute{e}t}(GL_n) = \hat{\mathbb{Z}}$$

$$SL_n \longrightarrow PGL_n \text{ étale} \iff (n, \mathrm{char}(k)) = 1$$

$$R = \{\pm e_i \pm e_j \mid 1 \leq i < j \leq n\}$$
$$\cup \{2e_i \mid 1 \leq i \leq n\}$$

$$\Delta = \{e_1 - e_2, \dots, e_{n-1} - e_n, 2e_n\}$$