

# The additive group $\mathbb{G}_a$ & unipotent Groups.

Notation. Let  $k = \bar{k}$  be a field

- Algebraic groups are assumed to be affine
- Denote by  $\underline{\mathbf{Alg}}$  the category of  $k$ -algebras
- Vector spaces are assumed to be finite-dimensional

Let  $R$  be an arbitrary ring. Recall that  $r \in R$  is called unipotent if  $1+r$  is nilpotent.

Let  $V$  be a  $k$ -vector space and let  $R = \text{End}_k(V)$ . Then  $r \in R$  is unipotent, iff  $\exists$  basis of  $V$  s.t.  $r = \begin{pmatrix} 1 & * & * \\ 0 & \ddots & * \\ 0 & \dots & 1 \end{pmatrix}$

Prop'n. Let  $\mathcal{G}$  be an abstract group that acts on a  $k$ -vector space  $V$  via unipotent endomorphisms. Then there exists a  $0 \neq v \in V$ , s.t.  $v \in V^{\mathcal{G}}$ .

Cor. In the situation above, there exists a basis of  $V$  s.t. the action of  $\mathcal{G}$  on  $V$  factors through  $U_n = \left\{ \begin{pmatrix} 1 & * & * \\ 0 & \ddots & * \\ 0 & \dots & 1 \end{pmatrix} \in \text{GL}_n(k) \right\}$ .

Def'n. An alg. grp  $\mathcal{G}$  is called unipotent, if every nonzero representation of  $\mathcal{G}$  has a non-zero fixed vector.

(If  $V$  is a  $k$ -vector space, we have

$$V^{\mathcal{G}} := \left\{ v \in V \mid g \cdot v_R = v_R \in V \otimes_k R \quad \forall R \in \underline{\mathbf{Alg}} \quad \forall g \in \mathcal{G}(R) \right\}$$

Def'n. Let  $(r, V)$  be a rep'ln of an alg. grp  $\mathcal{G}$ . Then  $(r, V)$  is called unipotent, if  $\exists$  basis of  $V$  s.t.  $r(\mathcal{G}) \subset U_n$  as a subgroup scheme.

Example.  $U_n(R) = \left\{ \begin{pmatrix} 1 & * & * \\ 0 & \ddots & * \\ 0 & \dots & 1 \end{pmatrix} \in \text{GL}_n(\mathbb{R}) \right\} \quad \forall R \in \underline{\mathbf{Alg}}$  is represented by

$$\text{Spec}(\mathbb{k}[T_{ij}, 1 \leq i < j \leq n]).$$

$$g_{ij}: \mathfrak{g}_a \longrightarrow \mathfrak{U}_n, \quad \begin{matrix} & (i,j) \text{th entry} \\ & \downarrow \text{1st row} \\ t & \longmapsto \begin{pmatrix} 1 & & & \\ & \ddots & & \\ & & 1 & \\ & & & 0 \end{pmatrix} \end{matrix}$$

Prop 4. An alg grp  $\mathfrak{G}$  is unipotent iff every representation  $(\rho, V)$  of  $\mathfrak{G}$  is unipotent.

Proof. [Hilf], Prop 4.3.

$$\mathcal{O}(\mathfrak{G})$$

Let  $A$  be a Hopf algebra.

Def'n. We call  $A$  connected, if there exists a filtration of

$$\mathbb{k}\text{-subspaces } \mathbb{k} = C_0 \subset C_1 \subset \dots \text{ s.t. } \bigcup_{r \in \mathbb{N}} C_r = A$$

$$\& m^*(C_r) \subset \bigoplus_{i=0}^r C_i \otimes_{\mathbb{k}} C_{r-i}$$

Let  $V$  be a fin-dim  $\mathbb{k}$ -vector space, and let  $\mathfrak{G}$  be an alg grp.

Def'n. A  $\overset{\mathcal{O}(\mathfrak{G})}{\text{comodule}}$  structure on  $V$  is the datum of a morphism  
 $\rho: V \rightarrow V \otimes_{\mathbb{k}} \mathcal{O}(\mathfrak{G})$  s.t.

$$(\text{id}_V \otimes m^*) \circ \rho = (\rho \otimes \text{id}_{\mathcal{O}(\mathfrak{G})}) \circ \rho$$

$$(\text{id}_V \otimes \varepsilon^*) \circ \rho = \text{id}_V,$$

Indeed: We have a bijection

$$\{ \text{rep's } \mathfrak{G} \text{ on } V \} \xleftrightarrow{\sim} \{ \mathcal{O}(\mathfrak{G})\text{-comodule structures on } V \}.$$

Then. Let  $\mathcal{G}$  be an affine algebraic group. Then the following are equivalent:

- $\mathcal{G}$  is unipotent.

(b)  $\mathcal{G}$  admits a closed immersion in  $\mathbb{U}_n$  for some  $n$ .

(c) the Hopf algebra  $\mathcal{O}(\mathcal{G})$  is cocompact.

prof. (a)  $\Rightarrow$  (b). We want to find a morphism  $\mathcal{G} \hookrightarrow \mathbb{U}_n$ .

$\Rightarrow$  It is sufficient to find a faithful rep'n of  $\mathcal{G}$  that is also unipotent.

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Indeed, the multiplication  $m: \mathcal{G} \times \mathcal{G} \rightarrow \mathcal{G}$  induces a comodule structure on  $\mathcal{O}(\mathcal{G})$ . One shows that the associated representation is faithful (we get  $\mathcal{O}(\mathbb{G}_L) \rightarrow \mathcal{O}(\mathcal{G})$ ).

$\Rightarrow$  this rep'n yields a closed immersion of  $\mathcal{G}$  into some  $\mathbb{U}_n$ .

(b)  $\Rightarrow$  (c) It is enough to show that the Hopf algebra that corresponds to  $\mathcal{O}(\mathbb{U}_n)$  is cocompact, i.e.  $k[\bar{T}_{ij}, i < j]$  together with

$$m^*(\bar{T}_{ij}) = \bar{T}_{ij} \otimes 1 + 1 \otimes \bar{T}_{ij} + \sum_{i < l < j} \bar{T}_{il} \otimes \bar{T}_{lj} \text{ is cocompact.}$$

We assign weights  $w_{ij} = j-i$  to  $\bar{T}_{ij}$ . Now let

$C_r$  be the subspace of  $\mathcal{O}(\mathbb{U}_n)$  generated by all monomials with weight  $\leq r$ .

In particular,  $C_0 = k$ ,  $\bigcup C_r = \mathcal{O}(\mathbb{U}_n)$ .

□

Cor. An alg grp  $\mathcal{G}$  is unipotent iff it admits a faithful unipotent

representation.

proof. See  $(a) \Rightarrow (b)$ .

Cor. Subgroups, quotients & extensions of unipotent groups are unipotent.

Cor. Every alg group contains a largest smooth connected normal unipotent subgroup.

Propn An alg group is unipotent iff it has a composition series whose quotients are subgroups of the additive group.

prof " Idea : Use the composition series of  $\mathbb{G}_a$  seen this morning.

"  $\Leftarrow$  follows as unipotence is stable under extension.

### Subgroups of $\mathbb{G}_a$

char 0.

Idea: All finite subgroups of  $\mathbb{G}_a(k)$  are trivial.

$\Rightarrow \mathbb{G}_a$  has only the trivial subgroup p.

char p.

•  $\alpha_p$  represented by  $\text{Spec}(k[T]/T^p)$

$\alpha_{p^n}(k) = \{x \in k \mid x^p = 0\}$  are subgroups of  $\mathbb{G}_a$  if

char  $k \neq 0$ .

•  $(f_i, t)$  represented by  $\text{Spec}(k[T]/T^{p-i})$ , & are the basic building blocks

$$\coprod_{x \in \mathbb{F}_{p^n}} S_i \quad \text{where } S_i = \{x \in k \mid x^{p^i} = 0\}$$