Algebraic Group actions

Manuel Hoff

Fix an algebraically closed field k.

Notation 0.1. We use the following notation:

- The terms "algebraic group/scheme" and "algebra" mean "k-(group) scheme of finite type" and "finitely generated k-algebra".
- We denote by Alg the category of algebras.
- We turn $\mathsf{Alg}^{\mathrm{op}}$ into a site by equipping it with the flat topology. A generating class of coverings is given by all finite families $(R \to R_i)_{i \in I}$ of maps in Alg such that the induced map $R \to \prod_{i \in I} R_i$ is faithfully flat.
- We will not distinguish between an algebraic scheme and its associated sheaf on $\mathsf{Alg}^{\mathrm{op}}$.

1 Recollection of (set-theoretic) group actions

Let G be an (abstract) group.

Definition 1.1. Let X be a set. A *group action of G on X* is a map of sets

$$G \times X \to X, \qquad (g, x) \mapsto g.x$$

that satisfies the following properties:

- $1.x = x$ for all $x \in X$.
- $(gh).x = g.(h.x)$ for all $g, h \in G$ and $x \in X$.

Giving a group action of G on X is equivalent to giving a map of groups $G \to S_X$ where S_X denotes the symmetric group on the set X. A set together with a group action by G is also called a *G-set*.

Definition 1.2. Let X be a G-set and let $x \in X$. Associated to x we have the following data:

- The subgroup
$$\mathrm{Stab}_G(x) \coloneqq \{g \in G \mid g.x = x\} \subseteq G$$
is called the *stabilizer of x in G*.
- The subset
$$G.x \coloneqq \{g.x \mid g \in G\} \subseteq X$$
is called the *orbit of x under the action of G*.

We have the following elementary fact:

Lemma 1.3. *Let X be a G-set and let $x \in X$. Then the map*

$$G/\operatorname{Stab}_G(x) \to G.x, \qquad g\operatorname{Stab}_G(x) \mapsto g.x$$

is a well-defined bijection.

2 Algebraic group actions

Let G be an algebraic group.

Definition 2.1. Let X be an algebraic scheme. An *algebraic group action of G on X* is a map of algebraic schemes

$$G \times X \to X, \qquad (g, x) \mapsto g.x$$

such that for every $R \in \mathsf{Alg}$ the induced map on points $G(R) \times X(R) \to X(R)$ is an action of the (abstract) group $G(R)$ on the set $X(R)$. An algebraic scheme together with an algebraic group action by G is also called an *algebraic G-scheme*.

The goal is now to define stabilizers and orbits for algebraic group actions. For the rest of the section, fix an algebraic G-scheme X and a k-rational point $x \in X(k)$ and denote the action map $G \to X$, $g \mapsto g.x$ by a_x.

2.1 Stabilizers

Definition 2.2. The *stabilizer of x in G* is the subpresheaf (of groups) $\mathrm{Stab}_G(x) \subseteq G$ that is defined by

$$\mathrm{Stab}_G(x)(R) \coloneqq \mathrm{Stab}_{G(R)}(x) \subseteq G(R)$$

for all $R \in \mathsf{Alg}$. Note that we (maybe confusingly) use the same notation for x and its image in $X(R)$.

Lemma 2.3. *$\mathrm{Stab}_G(x) \subseteq G$ is a (closed) algebraic subgroup.*

Proof. We have a pullback diagram

$$\begin{array}{ccc} \mathrm{Stab}_G(x) & \longrightarrow & G \\ \big\downarrow & & \big\downarrow{\scriptstyle a_x} \\ \mathrm{Spec}(k) & \xrightarrow{x} & X \end{array}$$

implying that $\mathrm{Stab}_G(x)$ is an algebraic scheme as desired. □

2.2 Orbits

Definition 2.4. The *orbit of x under the action of G* is the subsheaf $G.x \subseteq X$ that is given as the (sheaf-theoretic) image of the map $a_x \colon G \to X$.

More concretely, for an algebra R, the subset $(G.x)(R) \subseteq X(R)$ consists of all $y \in X(R)$ such that there exists a faithfully flat map of algebras $R \to R'$ and a group element $g \in G(R')$ such that $g.x = y$ in $X(R')$.

With this definition the following result is formal:

Lemma 2.5. *The map $a_x \colon G \to G.x$ induces an isomorphism $G/\mathrm{Stab}_G(x) \to G.x$ (where the quotient is the sheaf quotient).*

We now want to show that the definition of $G.x$ is well-behaved. To do this we will need the generic flatness theorem:

Theorem 2.6 (Generic flatness)**.** *Let $f \colon X \to Y$ be a map of algebraic schemes and assume that Y is reduced. Then there exists a (set-theoretically) dense open subscheme $U \subseteq Y$ such that the induced map $f^{-1}(U) \to U$ is flat.*

Lemma 2.7. *Suppose that G is smooth. Then $G.x \subseteq X$ is a locally closed subscheme and the map $a_x \colon G \to G.x$ is faithfully flat.*

Proof. Let $Z \subseteq X$ be the scheme-theoretic image of a_x (or equivalently the closure of the image of the underlying map of topological spaces equipped with the reduced subscheme structure). Then Z is stable under the G-action on X and the map $a_x \colon G \to Z$ is dominant.

As Z is reduced we may apply the generic flatness theorem to obtain a dense open subscheme $U \subseteq Z$ such that $a_x \colon a_x^{-1}(U) \to U$ is flat. As a_x is dominant (when corestricted to Z) we see that $a_x^{-1}(U) \neq \emptyset$. Shrinking U we thus may even assume that $a_x \colon a_x^{-1}(U) \to U$ is faithfully flat.

For every $g \in G(k)$ we have a commutative square

$$\begin{array}{ccc} a_x^{-1}(U) & \xrightarrow{\quad a_x \quad} & U \\ \big\downarrow{\scriptstyle g} & & \big\downarrow{\scriptstyle g} \\ g \cdot a_x^{-1}(U) = a_x^{-1}(g.U) & \xrightarrow{a_x} & g.U \end{array}$$

where the vertical maps are isomorphisms. Thus also the map $a_x \colon g \cdot \alpha^{-1}(U) \to g.U$ is faithfully flat. Taking the union over all g we see that also

$$a_x \colon G = \bigcup_{g \in G(k)} g \cdot a_x^{-1}(U) \to \bigcup_{g \in G(k)} g.U \eqqcolon V$$

is faithfully flat. As a faithfully flat map of algebraic schemes is in particular surjective as a map of sheaves, we have $V = G.x$. This proves the claim (V is an open subscheme of Z and Z is a closed subscheme of X). □

Remark 2.8. We make the following remarks.

- Without the smoothness assumption on G the above proof doesn't work anymore because Z may fail to be reduced. However the claim of the Lemma is still true (but to prove this requires more work).

- $G.x$ is always equidimensional (because G acts on it and this action is transitive on k-rational points). If G is smooth then also $G.x$ is smooth (because $a_x\colon G \to G.x$ is faithfully flat).

Lemma 2.9 (Closed orbit Lemma)**.** *Suppose that $G.x$ is of minimal dimension among the $G.y$ for $y \in X(k)$. Then $G.x \subseteq X$ is a closed subscheme.*

Proof. As before, let $Z \subseteq X$ be the scheme-theoretic image of a_x. We need to show that $G.x = Z$. So suppose that this is not the case.

Then there exists $y \in Z(k) \setminus (G.x)(k)$. The orbit $G.y \subseteq Z$ is disjoint from $G.x$, implying that $\dim G.y < \dim G.x$ and yielding a contradiction. □

2.3 Examples

Let's end with two examples:

Example 2.10. Suppose that k is of positive characteristic p. Let G be the algebraic group defined by

$$G(R) \coloneqq \{(g,t) \in R^\times \times R \mid t^p = 0\}$$

with group structure

$$(g,t) \cdot (g',t') \coloneqq (gg', gt' + t).$$

In other words, G is a semidirect product of $\mathbf{G}_m$ and α_p (for the action of $\mathbf{G}_m$ on α_p given by multiplication). Let $X \coloneqq \mathbf{A}^1$ and define an action of G on X by

$$(g,t).x \coloneqq gx + t.$$

One can verify that this is indeed a group action. There are the following two orbits:

- One orbit is given by $D(x) \subseteq \mathbf{A}^1$. The stabilizer of the representative $1 \in D(x)(k)$ is given by

  $$\operatorname{Stab}_G(1)(R) = \{(g,t) \mid g + t = 1\}$$

  and is isomorphic to μ_p via the map

  $$\mu_p \to \operatorname{Stab}_G(1), \qquad g \mapsto (g, 1-g).$$

- The other orbit is given by $V(x^p) \subseteq \mathbf{A}^1$ (and is not reduced). The stabilizer of its only point $0 \in D(x)(k)$ is given by

  $$\operatorname{Stab}_G(0)(R) = \{(g,t) \mid t = 0\}$$

  and is clearly isomorphic to $\mathbf{G}_m$.

Example 2.11. Let $G \coloneqq \mathrm{GL}_2$ and $X = \mathbf{P}^1$ (so that $\mathbf{P}^1(R)$ is the set of all rank 1 direct summands L of R^2). Then there is a natural action of G on X (elements in $\mathrm{GL}_2(R)$ can be considered as automorphisms of R^2).

This action has only one orbit and the stabilizer of the representative $[1,0] \in \mathbf{P}^1(k)$ is given by

$$\operatorname{Stab}_G([1,0])(R) = \left\{ \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \,\middle|\, c = 0 \right\}.$$