Algebraic Group actions

Manuel Hoff

Fix an algebraically closed field k.

Notation 0.1. We use the following notation:

- The terms "algebraic group/scheme" and "algebra" mean "k-(group) scheme of finite type" and "finitely generated k-algebra".
- We denote by Alg the category of algebras.
- We will not distinguish between an algebraic scheme and its associated functor $\mathsf{Alg} \to \mathsf{Set}$.

1 Recollection of (set-theoretic) group actions

Let G be an (abstract) group.

Definition 1.1. Let X be a set. A *group action of G on X* is a map of sets

$$G \times X \to X, \qquad (g, x) \mapsto g.x$$

that satisfies the following properties:

- $1.x = x$ for all $x \in X$.
- $(gh).x = g.(h.x)$ for all $g, h \in G$ and $x \in X$.

Giving a group action of G on X is equivalent to giving a map of groups $G \to S_X$ where S_X denotes the symmetric group on the set X. A set together with a group action by G is also called a *G-set*.

Definition 1.2. Let X be a G-set and let $x \in X$. Associated to x we have the following data:

- The subgroup
$$\mathrm{Stab}_G(x) \coloneqq \{g \in G \mid g.x = x\} \subseteq G$$
is called the *stabilizer of x in G*.
- The subset
$$G.x \coloneqq \{g.x \mid g \in G\} \subseteq X$$
is called the *orbit of x under the action of G*.

We have the following elementary fact:

Lemma 1.3. *Let X be a G-set and let $x \in X$. Then the map*

$$G/\mathrm{Stab}_G(x) \to G.x, \qquad g\,\mathrm{Stab}_G(x) \mapsto g.x$$

is a well-defined bijection.

2 Algebraic group actions

Let G be an algebraic group.

Definition 2.1. Let X be an algebraic scheme. An *algebraic group action of G on X* is a map of algebraic schemes

$$G \times X \to X, \qquad (g, x) \mapsto g.x$$

such that for every $R \in \mathsf{Alg}$ the induced map on points $G(R) \times X(R) \to X(R)$ is an action of the (abstract) group $G(R)$ on the set $X(R)$. An algebraic scheme together with an algebraic group action by G is also called an *algebraic G-scheme*.

The goal is now to define stabilizers and orbits for algebraic group actions. For the rest of the section, fix an algebraic G-scheme X and a k-valued point $x \in X(k)$ and denote the action map $G \to X$, $g \mapsto g.x$ by a_x.

2.1 Stabilizers

Definition 2.2. The *stabilizer of x in G* is the subfunctor (of groups) $\operatorname{Stab}_G(x) \subseteq G$ that is defined by

$$\operatorname{Stab}_G(x)(R) \coloneqq \operatorname{Stab}_{G(R)}(x) \subseteq G(R)$$

for all $R \in \mathsf{Alg}$. Note that we (maybe confusingly) use the same notation for x and its image in $X(R)$.

Lemma 2.3. *$\operatorname{Stab}_G(x) \subseteq G$ is a (closed) algebraic subgroup.*

Proof. We have a pullback diagram of functors $\mathsf{Alg} \to \mathsf{Set}$

$$\begin{array}{ccc} \operatorname{Stab}_G(x) & \longrightarrow & G \\ \downarrow & & \downarrow{\scriptstyle a_x} \\ \operatorname{Spec}(k) & \xrightarrow{x} & X \end{array}$$

implying that $\operatorname{Stab}_G(x)$ is an algebraic scheme as desired. □

2.2 Orbits

We now also want to define orbits for algebraic group actions. Before we do this we need one definition and one theorem that we treat as a blackbox.

Definition 2.4. A map of algebraic schemes $f\colon X \to Y$ is called *surjective* if the induced map on k-valued points $X(k) \to Y(k)$ is surjective (or equivalently if the underlying map of topological spaces $|X| \to |Y|$ is surjective).

Theorem 2.5 (Generic flatness)**.** *Let $f\colon X \to Y$ be a dominant map of algebraic schemes. Then there exists a (set-theoretically) dense open subscheme $U \subseteq Y$ such that the induced map $f^{-1}(U) \to U$ is surjective.*

Now we can state and prove our main result.

Lemma 2.6. *Suppose that G is smooth. Then there exists a unique locally closed subscheme $G.x \subseteq X$ with the following properties:*

- *The map $a_x\colon G \to X$ factors over $G.x$.*
- *The resulting map $a_x\colon G \to G.x$ is surjective (in the sense that it is surjective on k-valued points).*
- *$G.x$ is reduced.*

Proof. Let $Z \subseteq X$ be the closure of the image of the map of topological spaces $|a_x|\colon |G| \to |X|$, equipped with the reduced subscheme structure. As G is reduced the map a_x factors over $Z \subseteq X$ and the resulting map $a_x\colon G \to Z$ is dominant.

We now claim that Z is stable under the G-action on X, i.e. that the action map $G \times X \to X$ restricts to a map $G \times Z \to Z$. As $G \times Z$ is reduced it suffices to "show this on k-valued points" (meaning it suffices to show that $g.z \in Z(k)$ for all $g \in G(k)$ and $z \in Z(k)$). To see this, fix $g \in G(k)$ and consider the commutative diagram

$$\begin{array}{ccc} G & \xrightarrow{a_x} & X \\ \downarrow{\scriptstyle g} & & \downarrow{\scriptstyle g} \\ G & \xrightarrow{a_x} & X \end{array}$$

where we note that the vertical maps are isomorphisms. As Z is defined as the closure of the (topological) image of either of the horizontal maps, this implies that $g.Z = Z$.

Now we apply the generic flatness theorem to $a_x\colon G \to Z$ to obtain an open subscheme $U \subseteq Z$ such that the map $a_x\colon a_x^{-1}(U) \to U$ is surjective.

Now, for every $g \in G(k)$, we have a commutative diagram

$$\begin{array}{ccc} a_x^{-1}(U) & \xrightarrow{\quad a_x \quad} & U \\ \downarrow{\scriptstyle g} & & \downarrow{\scriptstyle g} \\ g \cdot a_x^{-1}(U) = a_x^{-1}(g.U) & \xrightarrow{a_x} & g.U \end{array}$$

where again the vertical maps are isomorphisms. Thus also the map $a_x\colon g\cdot \alpha^{-1}(U) \to g.U$ is surjective. Taking the union over all g we see that also

$$a_x\colon G = \bigcup_{g\in G(k)} g\cdot a_x^{-1}(U) \to \bigcup_{g\in G(k)} g.U =: V$$

is surjective.

Thus we see that we can take $G.x := V$ (this is a locally closed subscheme of X because it is an open subscheme of Z and Z is a closed subscheme of X). The uniqueness assertion is clear because the reduced locally closed subscheme $G.x \subseteq X$ is determined by its set of k-valued points $(G.x)(k)$ which is determined by the second assumption. □

Remark 2.7. We make the following remarks.

- When G is not smooth then the map $a_x\colon G \to X$ may fail to factor over Z. However there still exists a sensible definition of orbit, but this orbit may not be reduced anymore.
- $G.x$ is always equidimensional (because G acts on it and this action is transitive on k-rational points).

Lemma 2.8 (Closed orbit Lemma)**.** *Suppose that $G.x$ is of minimal dimension among the $G.y$ for $y \in X(k)$. Then $G.x \subseteq X$ is a closed subscheme.*

Proof. Let $Z \subseteq X$ be defined as in the proof of the previous Lemma. We need to show that $G.x = Z$. So suppose that this is not the case.

Then there exists $y \in Z(k)\backslash(G.x)(k)$. The orbit $G.y \subseteq Z$ is disjoint from $G.x$, implying that $\dim G.y < \dim G.x$ and yielding a contradiction. □

2.3 Examples

Let's end with two examples:

Example 2.9. Let $G := \mathrm{GL}_2$ and $X = \mathbf{P}^1$ (so that $\mathbf{P}^1(R)$ is the set of all rank 1 direct summands L of R^2). Then there is a natural action of G on X (elements in $\mathrm{GL}_2(R)$ can be considered as automorphisms of R^2).

This action has only one orbit and the stabilizer of the representative $[1,0] \in \mathbf{P}^1(k)$ is given by

$$\mathrm{Stab}_G\big([1,0]\big)(R) = \Big\{ \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \,\Big|\, c = 0 \Big\}.$$

Example 2.10. Suppose that k is of positive characteristic p. Let $G := \mu_p$ and $X := \mathbf{A}^1$ and define an action of G on X by

$$g.x := g\cdot x.$$

Let $x := 1 \in \mathbf{A}^1(k)$. Then the topological image of the action map $a_x\colon G \to X$ is just given by the single point x again (or more precisely the closed point of $|X|$ that corresponds to x). But the action map $a_x\colon G \to X$ does not factor over $V(t-1) \subseteq X$.

To see this, consider $R := k[\varepsilon]/(\varepsilon^2) \in \mathsf{Alg}$ and the R-valued point $g := 1+\varepsilon \in G(R)$. Then we have

$$a_x(g) = (1+\varepsilon)\cdot 1 = 1+\varepsilon \notin V(t-1)(R).$$