Workshop Linear Algebraic Groups

Darmstadt, Sep. 27 – Oct. 8, 2021

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We will learn in condensed form about the wonderful theory of linear algebraic groups. Topics include:

1. Algebraic Groups and basic constructions with algebraic groups
2. Multiplicative Groups, unipotent groups, solvable groups
3. Reductive groups, Maximal tori, Borel subgroups, parabolic subgroups
4. Classification of reductive groups via based root data

The level should be suitable for all students with one semester of Algebraic Geometry background.

All schemes and group schemes will be of finite type over an algebraically closed field.

Organization

The talks will take place in the morning for approximately 3h every day. We will start on all days at 8:45.

All talks will be online. The Zoom coordinates are
Meeting-ID: 857 7156 8177
Kenncode: 349359

Talks by participants (60min each) are in boldface, Talks by the organizers are not.

In the afternoon, there will be optional, tutorium-style meetings with the possibility to discuss the material presented that day. If you are doing a talk yourself, you can also use this opportunity to ask questions about it.

Web page with current informations: https://crc326gaus.de/workshops-block-courses/

Program

All schemes/varieties/algebraic groups are defined over an algebraically closed field k.

Day 1 (Monday, Sept. 27th)**. T. Wedhorn: Algebraic Groups and basic constructions with algebraic groups**

1. Notion of group scheme as group object in the category of schemes, homomorphisms of group schemes, kernel of homomorphisms
2. Algebraic groups
3. Affine algebraic groups and Hopf algebras
4. Reduced algebraic groups are smooth, in characteristic 0 all group schemes are smooth
5. Surjectivity of homomorphisms
6. Many examples
7. Affine group schemes are linear, Rosenlicht's theorem
8. Representations of algebraic groups
9. Jordan decomposition

Day 2 (Tuesday, Sept. 28th).

1. **Manuel Hoff: Group schemes acting on schemes, orbits are locally closed, existence of closed orbits:** The goal is to show that orbits are locally closed and to prove the closed orbit lemma [Con, Thm 9.3.5.], see also [Spr, 2.3]. Very briefly recall group actions and orbits from abstract group theory. Define the relevant notion for algebraic groups ([Mil, 1.f]; also emphazise the difference between the schematic and k-rational definitions), state [Con, Thm 9.3.5.] and mention some of the examples and consequences in [Con][9.3]. Finally, prove [Con, Thm 9.3.5.] (note that the proof simplifies by our assumption that $k = \overline{k}$, so in particular k is separably closed).
2. **Annika Jäger: Homogeneous spaces and homogenity principle, smooth homogeneous spaces:** The goal is to prove [Spr, 5.3.2] and its corollary. Define homogenous spaces [Spr, 2.3.1] (note that Springer assumes an algebraic group to be reduced and hence smooth; define homogeneous spaces only in this case[1]). Mention [Spr, 5.1.6, 5.1.7] without proof, but try to explain its geometric meaning. Finally, prove [Spr, 5.3.2, 5.3.3].
3. Quotients, properties of surjective homomorphisms and their kernels, bijective homomorphisms are isomorphisms in characteristic 0
4. Composition series

[1] The notion of a homogeneous space for a non-reduced algebraic group is more complicated.

Day 3 (Wednesday, Sept. 29th)**.** Solvable groups

1. **Catrin Mair: Groups of multiplicative type = diagonalizable groups, character group of an algebraic group, equivalence between diagonalizable groups and finitely generated abelian groups, representation of diagonalizable groups**

   The goal is to explain multiplicative groups which are the same as diagonalizable groups because our ground field is algebraically closed ([Mil, 12.a-d]). Define the group $X^*(G)$ of characters of an algebraic group. Formulate and prove the classification [Mil, Thm 12.9]. Note that since our base field is algebraically closed, you can assume throughout that multiplicative type=diagonalizable, torus=split torus, Gal(k_s/k)=0 etc. You may mention however that all of the theory works if k is simply assumed to be separably closed. Explain the notion of a representation of an algebraic group.

   Show that if G is a diagonalizable group, then to endow a finite-dimensional vector space V with the structure of a representation of G is the same as to define as to define an $X^*(G)$-grading on V ([Mil, 12.13]).

2. **Jacob Burgi: Subgroups of the additive group, unipotent groups**

   After introducing the necessary terminology, prove [Mil, Thm. 14.5] and showcase some of its corollaries, in particular Cor. 14.6, 14.7, 14.8. Deduce that an algebraic group is unipotent if and only if it has a composition series whose quotients are isomorphic to subgroups of the additive group (use that this is the case for the group of unipotent upper triangular matrices). Explain [Mil, 14.24(a)]. Mention some of the examples in [Con, lecture 15].

3. Solvable groups, composition series of connected smooth solvable groups

4. Decomposition of (commutative) solvable groups.

Day 4 (Thursday, Sept 30th)**.** These two talks should be prepared jointly by two participants. The distribution of the material over the two talks presented here is just a recommendation. No knowledge about algebraic groups is required, however the level of the algebraic geometry involved is somewhat higher than in the rest of the seminar.

1. **Christopher Lang: Digression: Schemes of finite type and separated morphisms, Proper morphisms and valuation criteria I**
   Define morphisms of schemes of finite type, separated morphisms, and proper morphisms as in [GW]. Offer some geometric intuition for the latter two in the case of a scheme over a base field as being replacements for "Hausdorff" and "compact", respectively. Mention some permanence properties such as stability under composition or base change. Give some (but not all) proofs.

2. **Hendrik Petzler: Digression: Schemes of finite type and separated morphisms, Proper morphisms and valuation criteria II** Introduce valuation rings [Adic, 2.1–2.3, 2.6–2.11] (sketch only proofs), and describe the geometric shape of the spectrum of discrete valuation rings. State without proof the valuative criteria for proper and separated morphisms ([GW, 15.8–15.10], concentrating on the noetherian case) and offer some intuition in terms of (uniquely) "filling holes".

Day 5 (Monday, Oct. 4th)**.** Maximal tori, Borel subgroups, parabolic subgroups

1. **Lena Volk: Borel's fixed point theorem and applications** Formulate and prove Borel's fixed point theorem ([Con][20.2], see also [Mil][17.a] for a different proof) and its corollaries [Con][20.2.2 – 20.2.4].

2. **Can Yaylali: Maximal tori, Borel subgroups, parabolic subgroups, one-parameter subgroups**

Day 6 (Tuesday, Oct. 5th)**.** Reductive groups and root data

1. Unipotent radical, reductive and semisimple groups, examples, classification of reductive groups of rank 1, Weyl groups

2. **Till Rampe: Root data** Explain [Spr, 7.4.1, 7.4.2(c), 7.4.5 (first half)]. Define also what it means for a root datum to be reduced (last line before [Spr, 7.4.4]). Give examples of root data as in [Spr, 7.4.7]. Prove [Spr, 7.5.1].

Day 7 (Wednesday, Oct. 6th)**.** Root data attached to reductive groups, classification of reductive groups, Langlands correspondence

References

[Adic] T. Wedhorn: Adic Spaces, lecture notes, arXiv:1910.05934

[Bor] A. Borel: Linear Algebraic Groups, Springer Verlag

[Bou] N. Bourbaki: Lie groups and Lie algebras, IV – VI, Springer Verlag

[Con] B. Conrad: Algebraic Groups (lecture notes)

[GW] U. Grtz, T. Wedhorn: Algebraic Geometry I, Springer Verlag

[Mil] J. Milne: Algebraic Groups, Cambridge studies in advanced mathematics

[Spr] T.A. Springer: Linear Algebraic Groups, Birkhäuser Verlag