Homogeneous spaces and homogenity principle,
smooth homogeneous spaces

Workshop Linear Algebraic Groups
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Contents

1 Introduction

This script is completely based on the book "Linear Algebraic Groups" by T. A. Springer. We will focus on the definition of homogeneous spaces and the proof of our main theorem which can be found in chapter 5 of the mentioned book. Our goal is to understand homogeneous spaces and some of their properties. On our way to our main result, we will also find out some interesting properties of algebraic groups that will turn out to be very useful.

2 Definition of homogeneous spaces

In the following, we will assume all algebraic groups to be reduced and hence smooth. This means that in particular every algebraic group is a variety endowed with a group structure.

Definition 1. *Let X be a variety and let G be an algebraic group. If we have a morphism of varieties $a : G \times X \to X,\ a(g, x) = g.x$ such that $g.(h.x) = (g.h).x,\ e.x = x\ (g, h \in G, x \in X)$, we call X a G*-space.
Let X and Y be G-spaces. A morphism $\phi : X \to Y$ is called G-morphism *or* equivariant *if $\phi(g.x) = g.\phi(x),\ (g \in G, x \in X)$.*

Definition 2. *A variety X is a* homogeneous space for *G if it is a G-space on which G acts transitively, i.e. for every $x, y \in X$ we have a $g \in G$ such that $g.x = y$.*

Example 3. *A homogeneous space for G for which all the isotropy groups are trivial is called* principal homogeneous space *or* torsor. *An example of a torsor is G acting on itself by left (resp. right) translation:*

$$a : G \times G \to G,\ (g, h) \mapsto gh\ (\textit{resp.}\ (g, h) \mapsto hg^{-1}).$$

It is easy to see that the isotropy group of every $x \in G$ is trivial since the neutral element of a group is unique. Hence, G even acts simply transitive on G in this case, i.e. for all $x, y \in G$ there is a unique *$g \in G$ such that $g.x = y$.*

Example 4. *Let X be a variety and G be an algebraic group. In the prior talk we have seen that an orbit $G.x$ is a locally closed subset of X. Therefore, it has a variety structure induced by X. Since G is an algebraic group, it is clear that G acts transitively on $G.x$, so $G.x$ is a homogeneous space for G.*

3 Identity component

Recall 5. *In the lecture we defined a topological space X to be irreducible if there are no two non-trivial subsets of X such that their union equals X. In other words, X is irreducible if for two arbitrary non-empty open subsets of X the intersection of the two sets is non-empty.*

Definition 6. *The maximal irreducible subsets of a topological space Xin terms of set inclusion are called* (irreducible) componets *of X.*

Remark 7. *One can show that the irreducible components of a homogeneous space for an algebraic group G denoted by X coincide with the connected components of X. Hence, we will simply call them* components *of X. The proof of this remark is a simple corollary of Proposition 13 that we will show at the end of this section. Recall that a topological space is* connected *if it cannot be represented by the union of two disjoint open sets where none of them is the empty set. Hence, every irreducible set is connected.*

Lemma 8. *Let X be a topological space.*

i) A subset $A \subset X$ is irreducible if and only if its closure $\overline{A}$ is irreducible.

ii) Let Y be a topological space and let $f : X \to Y$ be a continuous map. If X is irreducible then its image $f(X)$ is also irreducible.

Lemma 9. *Let X be a topological space that is noetherian. Then X has finitely many irreducible components.*

Proposition 10. *Let G be an algebraic group.*

i) There is a unique component G^0 of G that contains the identity element of G.

ii) G^0 is a closed normal subgroup of G.

iii) G^0 is of finite index in G, i.e. there exist just finitely many left cosets of G^0 in G.

Proof. i) Let X and Y be components of G containing its identity element. Let μ be the morphism inducing the multiplication of the group structure of G. Since the product of irreducible spaces is irreducible, we get by Lemma 8 that $\mu(X \times Y) = XY$ and also its closure $\overline{XY}$ are irreducible. Additionaly, we know that $X \subset XY \subset \overline{XY}$, $Y \subset XY \subset \overline{XY}$ because both X and Y contain the identity element. The maximality of X and Y implies that $X = Y = XY = \overline{XY}$.
ii) From the proof of i) we get that $G^0 = G^0 G^0$ and hence G^0 is closed under multiplication. Let $i : G \times G \to G,\ g \mapsto g^{-1}$ be the morphism being part of the group structure of G. Since i is a homeomorphism, it is easy to see that $(G^0)^{-1}$ is also a component of G containing its identity element. So by i) we get that $(G^0)^{-1} = G^0$ which means that G^0 is closed under taking inverses. Hence, G^0 is a subgroup of G. It is also closed, since $G^0 = \overline{G^0}$ which we have seen in the proof of i). Moreover, since inner automorphisms define homeomorphisms, we get that gG^0g^{-1} is homeomorphic to G^0 and contains the identity element. By i) we get that $gG^0g^{-1} = G^0$ and thus G^0 is a normal closed subgroup of G.
iii) Since translations are homeomorphisms, all cosets $gG^0,\ (g \in G)$ must be components of G. We know that G, considered as a topological space, is noetherian. This is because G is a scheme of finite type over a field k. By Lemma 9, the algebraic group G has finitely many components. Thus, G^0 is of finite index in G. □

Definition 11. *By G^0 we denote the component of the algebraic group G containing the identity of G. We call G^0 the* identity component *of G. We know by Remark 7 that G^0 coincides with the connected component of G.*

Example 12. *The special orthogonal group SO_n is the identity component of the orthogonal group O_n.*

Proposition 13. *Let G be an algebraic group and let X be a homogeneous space for G.*

i) X is the disjoint union of its components.

ii) The components of X are open and closed.

iii) Each component of X is a homogeneous space for G^0.

Proof. Let X' be an orbit of G^0, i.e. $X' = G^0.x$ for an $x \in X$. By Proposition 10 iii) we know that

$$\bigcup_{i \in I} g_i G^0 = G \text{ for } g_i \in G,\ |I| < \infty.$$

Now by the associativity of the multiplication we get that

$$G.x = \bigcup_{i \in I} g_i G^0.x = \bigcup_{i \in I} (g_i.X').$$

Additionally, since G acts transitively on X, we have that $G.x = X$. Hence, it is $X = \bigcup_{i \in I}(g_i.X')$, i.e. X is the disjoint union of finitely many translates $g_i.X'$. Assume $g_i.X' \cap g_j.X' \neq \emptyset$. Then there exist $a, b \in G^0$ such that $a.g_i.x = b.g_j.x$ (using normality of G^0). Then we have for an arbitrary element $c \in G^0$ that

$$c.g_i.x = c.a^{-1}.a.g_i.x = c.a^{-1}.b.g_j.x = ca^{-1}b.g_j.x \in g_j.X'.$$

Hence, we have $g_i.X' = g_j.X'$. Using that G^0 is a normal group, which we know by Proposition 10 ii), we get that every $g_i.X'$ is a G^0-orbit since

$$g_i.X' = g_i.G^0.x = G^0.g_i.x = G^0.\tilde{x}$$

where we also used the associativity of multiplication. From the prior talk we know that there is a G^0-orbit that is closed. All the translates of this orbit then have to be closed too and hence all G^0-orbits are closed.
Since the map $a : G \times X \to X$, which makes X a G-space, is continuous and $G^0 \times \{x\}$ is irreducible in $G \times X$ for every $x \in X$, we get by Lemma 8 ii) that every G^0-orbit is irreducible in X. Therefore every $g_i.X'$ is irreducible in X. So we know that X is the disjoint union of finitely many closed irreducible sets $g_i.X'$.
What is left to show is that they are maximal. Let Y be irreducible and $g_i.X' \subset Y$. W.l.o.g. we also assume Y to be closed (using Lemma 8 i)). Then every intersection of Y with a closed set $g_i.X'$ is closed. Since X is covered by finitely many $g_i.X'$ we get that Y is the union of finitely many closed sets. In particular, we get $Y = g_i.X' \cup (\bigcup_{j \in I \setminus i} Y \cap g_j.X')$ which makes Y the union of two closed sets. By the assumption of Y being irreducible, we get $\bigcup_{j \in I \setminus i} Y \cap g_j.X' = \emptyset$ and hence, the $g_i.X'$ are the components of X. It is easy to see that there cannot be any further components of X. This proves that X is the disjoint union of its components.
To prove ii), it is left to show that the components of X are also open. This easily follows from i) and the fact that the components are closed. Every component is the intersection of finitely many open sets, i. e.

$$g_i.X' = \bigcap_{j \in I \setminus i} (g_j.X')^{\mathsf{c}},$$

and is therefore open.
To show iii), it suffices to show that G^0 acts transitively on the $g_i.X'$. This is true since G^0, as an algebraic group, acts transitively on itself.

□

4 Properties of dominant morphisms

We take a look at a theorem that is essential for our main theorem. Even if we will not prove it here, we will still talk about the geometric meaning of its proposition. But first we need some more definitions. For more information and the proofs of Theorem 17 and Corollary 18 see (Springer, 1998) chapter 5.1.

Definition 14. *A morphism $\phi : X \to Y$ of irreducible varieties is called* dominant *if $\phi(X)$ is dense in Y.*

Let k be an algebraically closed field and let X be an irreducible variety over k. Let U be an affine algebraic set contained in X We will denote the algebra of functions on U by $\mathcal{O}_X(U) = k[T_1, ..., T_n]/I(U)$ like we did in the lecture. So we can define $k(X)$ to be the quotient field of $\mathcal{O}_X(U)$. We can do that since $k(X)$ does not independ on the choice of U.
Now let $\phi : X \to Y$ be a dominant morphism of irreducible varieties. Then there is a corresponding algebra homomorphism

$$\phi^* : \mathcal{O}_Y(Y) \to \mathcal{O}_X(X),\ g \mapsto g \circ \phi.$$

This induces a morphism $k(Y) \to k(X)$. If we additionally have that ϕ is dominant, then the map $\mathcal{O}_Y(Y) \to \mathcal{O}_X(X)$ is injective. This is because if ϕ is dense in Y, then $I_Y(\phi(X)) = \{0\}$, since all principal open sets of Y have to intersect with $\phi(X)$ and for $g \in \mathcal{O}_Y(Y)$ we have

$$\phi^*(g) = 0 \in \mathcal{O}_X(X) \Leftrightarrow \phi^*(g) \in I(X) \Leftrightarrow g(\phi(X)) = 0.$$

Therefore also the morphism $k(Y) \to k(X)$ is injective and hence we can view $k(X)$ as a field extension of $k(Y)$.

Theorem 15. *Let X and Y be irreducible varieties and let $\phi : X \to Y$ be a dominant morphism. Define $r := \dim X - \dim Y$. There is a non-empty open set $U \subset X$ with the following properties:*

i) The restriction of ϕ to U is an open morphism $\phi|_U : U \to Y$.

ii) If Y' is an irreducible closed subvariety of Y and X' is an irreducible component of $\phi^{-1}Y'$ that intersects U, then $r = \dim X' - \dim Y'$. In particular, if $y \in Y$, any irreducible component of $\phi^{-1}y$ that intersects U has dimension r.

iii) If $k(X)$ is algebraic over $k(Y)$, then for all $x \in U$ the number of points in the fiber $\phi^{-1}(\phi(x))$ equals $[k(X) : k(Y)]_s$.

Corollary 16. *In the case of Theorem 15, we may replace i) by the following stronger property:*

i)' For a variety Z the restriction of ϕ to U induces an open morphism $(\phi|_U, id) : U \times Z \to Y \times Z$.

Theorem 15 ii) gives geometric interpretation of the transcendence degree of a quotient field $k(X)$. In fact, every fiber intersecting with the dense open set U has the same dimension. Moreover, the transcendence degree of $k(X)$ over $k(Y)$ coincides with the dimension of these fibers. In the case of $k(X)$ being algebraically closed over $k(Y)$ part ii) of Theorem 15 becomes uninteresting but part iii) gives us geometric interpretation of $[k(X) : k(Y)]_s$. Here, every fiber intersecting with the dense open set U contains the same number of points of X. This number coincides with the separabitity degree of $k(X)$ over $k(Y)$.

5 Properties of G-morphisms on homogeneous spaces

Lemma 17. *Let X be an irreducible variety. Then the simple points of X form a non-empty open subset of X.*

We already used this fact yesterday. For the proof, see Springer (1998) page 67.

Proposition 18. *Let G be a connected algebraic group and let X be a homogeneous space for G.*

i) The space X is irreducible and smooth. In particular, G is smooth.

ii) Let $\phi : X \to Y$ be an equivariant dominant homomorphism of homogeneous spaces for G. Then ϕ is separable if and only if the tangent map $d\phi_x$ is surjective for some $x \in X$. If this is the case, then $d\phi_x$ is surjective for alle $x \in X$.

Proof. We know by Remark 7 that G is irreducible. Define the map $G \to X,\ g \mapsto g.x$ which is surjective since G acts transitively on X. Then we get by Lemma 8 i) that X is irreducible. By Lemma 17 we can find a simple point x of X. For a fixed $g \in G$ the map $X \to X,\ x \mapsto g.x$ is a isomorphism for every $g \in G$. Hence, again by using that G acts transitively on X, we get that X is smooth. Since G is a homoheneous space for G (see Example 3), G is smooth. So we have another way of showing that G is smooth besides using it being reduced. For the proof of part ii) of this lemma, see Springer (1998) page 69. □

Theorem 19. *Let G be an algebraic group, let $\phi : X \to Y$ be an equivariant homomorphism of homogeneous spaces for G and define $r := \dim X - \dim Y$.*

i) For any variety Z the morphism $(\phi, id) : X \times Z \to Y \times Z$ is open.

ii) If Y' is an irreducible closed subvariety of Y and X' is an irreducible component of $\phi^{-1}Y'$, then $r = \dim X' - \dim Y'$. In particular, if $y \in Y$, then all irreducible components of $\phi^{-1}y$ have dimension r.

iii) The morphism ϕ is an isomorphism if and only if it is bijective and for some $x \in X$ the tangent map $d\phi_x : T_xX \to T_{\phi_x}Y$ is bijective.

Proof. In the first step we want to reduce the proof to the case where G is connected and X and Y are irreducible. This is the moment where the effort of proving Proposition 13 pays off. We get by i) and ii) of Proposition 13 that the components of X and Y form a finite open and closed cover of X resp. Y. Since ϕ is an equivariant homomorphism, we get that

$$\phi(g.X') = \phi(g.G^0.x) = gG^0.\phi(x)$$

which is a component of Y. So ϕ sends components of X to corresponding components of Y. In particular, ϕ is surjective on a component mapping to the corresponding component. Hence, open sets as well as irreducible closed subvarieties can all be considered in components of X and Y. So we can assume X and Y to be irreducible. By Proposition 13 iii) we know that the components are homogeneous spaces for G^0. Hence, we can assume G to have the same properties as G^0 in this case. Thus, following Remark 7, we get that G is connected. Additionally, we know that in this case ϕ is surjective and hence dominant.
Now we can use Theorem 15 and Corollary 16. Let U be the open non-empty set we get in Theorem 15 with the properties i)'-iii). We also know that $\bigcup_{g\in G} g.U = X$ where all the translates $g.U$ have the same properties as U since translations are homeomorphisms. Thus, i) follows from i)' of Corollary 17. Since (ϕ, id) is open on every set $U \times Z$ where the U are an open cover of X, ϕ has to be open on $X \times Z$.
Furthermore, ii) follows from ii) of Theorem 15. For an arbitrary $y \in Y$ any irreducible component of ϕ^{-1} has to intersect with a $g.U$ since the $g.U$ cover X because of G acting transitively on X. Hence, we directly get ii) by applying ii) of Theorem 15.
Finally, we will prove iii). It is clear that ϕ being an isomorphism implies that ϕ and also the tangent map $d\phi_x$ is bijective for all $x \in X$.
For the converse, assume that ϕ and $d\phi_x$ for some $x \in X$ are bijective. This implies that $k(X)$ is algebraic over $k(Y)$ because by ii) we have that $\dim(X) = \dim(Y)$. Since G is connected, we get from Proposition 18 ii) that ϕ is separable. This means that $k(X)$ is a separably generated extension of $k(Y)$. Moreover, by Theorem 17 iii) we get that $[k(X) : k(Y)]_s = 1$. So we have that $[k(X) : k(Y)] = 1$ and hence $k(X) = k(Y)$.
Now one can show that there is a non-empty open subset V of X such that $\phi(V)$ is open and ϕ induces an isomorphism of varieties $V \backsimeq \phi(V)$. We can assume X and Y to be affine since both have covers of open affine sets an later we will see that U can be viewed as included in one such cover. Since ϕ is birational we have $\mathcal{O}_X(X) = \mathcal{O}_Y(Y)[f_1, ..., f_r]$ where all f_i lie in $k(Y)$. Now take an element $f \in \mathcal{O}_Y(Y)$ such that $f \neq 0$ and $ff_i \in \mathcal{O}_Y(Y)$ for all $0 < i \leq r$. Then ϕ^* induces an isomorphism $\mathcal{O}_X(X)_f \cong \mathcal{O}_Y(Y)_f$. Hence we get that the open subset U of X we were looking for is $D_X(f)$ and $D_X(f) \cong \phi(D_X(f))$.
Again, we use that $X = \bigcup_{g\in G} g.V$ since G acts transitively on X. Then finally we have that

$$\phi(X) = \phi(\bigcup_{g\in G} g.V) = \bigcup_{g\in G} \phi(g.V) = \bigcup_{g\in G} g.\phi(V) \cong \bigcup_{g\in G} g.U = X$$

using that ϕ is an equivariant homomorphism. This proves iii).

□

Corollary 20. *Let $\phi : G \to G'$ be a surjective homomorphism of algebraic groups.*

i) It is $\dim G = \dim G' + \dim \ker \phi$.

ii) *ϕ is an isomorphism if and only if ϕ and the tangent map $d\phi_e$ are bijective where e is the identity element of G.*

Proof. As ϕ is a homomorphism of algebraic groups, it is a morphism of varieties which is also a group homomorphism. We will use that to view G and G' as homogeneous spaces for G. First, G can be viewed an a homogeneous space for G like we did in Example 3. For G', define a morphism of varieties

$$a : G \times G' \to G',\ (g, g') \mapsto g.g' := \phi(g)g'.$$

Since ϕ is a group homomorphism, we have $g.(h.g') = (g.h).g'$ as well as $e.g' = g'$ for all $g, h \in G, g' \in G'$. Since ϕ is surjective, we also get that G acts transitively. Now by definition, ϕ is also an equivariant homomorphism of homogeneous spaces for G. In fact for $g, h \in G$ we have

$$\phi(g.h) = \phi(gh) = \phi(g)\phi(h) = g.\phi(h).$$

Hence, we can apply Theorem 19.
Take $e' \in G'$ the identity element of G'. By Theorem 19 ii) we get that all irreducible components of $\phi^{-1}(e') = \ker(\phi)$ have dimension $r = \dim G - \dim G'$. The definition of the dimension then already gives us that $\dim \ker(\phi) = \dim G - \dim G'$.
It is clear that ϕ being an isomorphism implies that ϕ and the tangent map $d\phi_e$ are bijective. We get the converse by applying Theorem 19 iii). □

An example of a surjective homomorphism of algebraic groups that is bijective but not an isomorphism can be found in Springer (1998) page 23.

References

T.A. Springer. *Linear Algebraic Groups*. Birkhäuser Boston, 2nd edition, 1998.