Groups of multiplicative type
- base field \( k \) algebraically closed
- all algebraic groups affine: \( G = \text{Spec} A \)

**Definition**
An algebraic group is of multiplicative type if it becomes diagonalizable over the separable closure of the base field \( k \). (i.e. \( G_{k^{sep}} \cong \bigotimes \mathbb{A}_k \text{diag.} \))

\( k \text{ alg. closed} \Rightarrow \text{ multiplicative type} = \text{ diagonalizable} \)

**Hence:**

- **Diagonalizable Groups**
  - \( V \) always finite dimensional \( k \)-vector space

**(A) Characters & diagonalizable representations**

Recall A.1.
A morphism \( r: G \to GL_V \cong GL_n \) is called a representation.

**Definition A.2.**
A 1-dim. representation (i.e. \( \dim V = 1 \))

\( \chi: G \to GL_1 \cong \mathbb{G}_m \)

is called a character of \( G \).

**Remark A.3.** Group-like elements

\( \chi: G \to \mathbb{G}_m \) (k-scheme morphism resp. \( m, e, i \))

\( \chi: k[C(T^{-1})] \to A \)

(i.e. given by \( T \mapsto a \))

\( a \).
Obtain commutative group structure by \( \mathcal{X} + \mathcal{Z} : \text{GL}(R) \to R^x \),
\( g \mapsto \chi(g) \mathcal{X}(g) \)
\( \chi(g) = \text{abelian group of } \mathcal{X} \text{ on } \mathcal{G} \)

Example A.4. Diagonal representations

Let \( V = \mathbb{C}^n \)

1) \( \chi : \mathcal{G} \to \text{GL}(n) \) defines representation \( \rho \) of \( \mathcal{G} \) on \( V \) by

\[
\rho : \mathcal{G} \to \text{GL}(n) \quad (g) \to \begin{pmatrix} \chi_1(g) & \cdots & \chi_n(g) \end{pmatrix} \in \text{GL}(n) \]

i.e. \( \rho(g) \cdot v = \chi(g) 
\text{V} \quad (\forall g \in \mathcal{G}, \forall v \in \mathcal{V}) \)

"\( \mathcal{G} \) acts on \( \mathcal{V} \) through \( \chi" \)

2) More generally: \( \chi_1, \ldots, \chi_n \) \( (\forall i = 1 \text{ allowed}) \)

\[
\bigoplus_{i=1}^{n} \chi_i = \rho : \mathcal{G} \to \text{GL}(n) \quad (g) \to \begin{pmatrix} \chi_1(g) & \cdots & \chi_n(g) \end{pmatrix} \in \text{GL}(n) \]

Definition A.5: eigenspace

\( (V, \chi) : \mathcal{G} \to \text{GL}(n) \), \( Z \in \chi(\mathcal{G}) \)

If there exists \( \forall v \in \mathcal{V} \) st. \( \forall g \in \mathcal{G} \)

\[
\rho(g) \cdot v = \chi(g) 
\text{V} \quad (\forall v \in \mathcal{V}) \]

then "\( \mathcal{G} \) acts on \( \mathcal{V} \) through \( \chi" \)

\( V_\chi = \{ v \in \mathcal{V} \mid \rho(g) \cdot v = \chi(g) 
\text{V} \quad (\forall v \in \mathcal{V}) \}

"eigenspace for \( \mathcal{G} \) with character \( \chi" \)
Example: \( r = \bigoplus_{i_1 = 1}^n X_i \) (as in A.4.2) \( \Rightarrow \) \( e_i \in k^n \) is eigenvector of \( r \) corresponding to \( X_i \) \( \Rightarrow \) \( V = \bigoplus_{\chi \in \chi(G)} V_\chi \) (almost all \( V_\chi \) zero)

Question: Let \((V_1, r) : G \rightarrow GL(V)\) be arbitrary under which assumption there exists a basis for \( V \) s.t. \( r(G_1) \in D_n \cong GL_n(k) \)

\( \Leftrightarrow \) \( r = \bigoplus_{i=1}^n X_i \) for some \( X_i \in \chi(G) \)

Answer: \( G \) diagonalizable group

In fact:
Theorem: Equivalent:
a) \( G \) diagonalizable
b) every finite dimensional representation is direct sum of characters
c) for every representation \((V_1, r)\) of \( G : V = \bigoplus_{\chi \in \chi(G)} V_\chi \)

(B) Diagonalizable Groups
Let \( N \) be finitely generated abelian group.
Then \( kCN_1 : k\)-vector space with basis \( N \)
(i.e. \( kCN_1 \exists x = \sum_{i \in N} a_i \cdot \overline{n_i} \) \( a_i \in k, n_i \in N \) )

1) is finitely generated \( k \)-algebra by extending multiplication of \( N \)
2) is Hopf algebra by setting
\( \mu(x) = n \otimes \overline{n_i} \), \( e(x) = 1 \), \( \Delta(x) = x^{-1} \)

Example B.1.
1) \( N = Z : k[CN] \exists x = \sum_{a \in \mathbb{Z}} a \cdot \overline{i} \), \( a \in k, a_i = 0 \) for \( a_i \neq 0 \) \( a \in \mathbb{Z} \)

\( \Rightarrow k[CN] \cong k[T^{\mathbb{Z}}] \) as Hopf algebras

2) \( N = \mathbb{Z} / n \mathbb{Z} : k[CN / n] \exists x = \sum_{a_i \in \mathbb{Z} \mod n} a_i \cdot \overline{i} \Rightarrow k[CN / n] \cong k[T^\mathbb{Z} / n] \)
Lemma B.2.
a) $N_1 \times N_2$ fg. ab. gr. : $kCN_1 \times \times kCN_2$ as $k$-algebras
b) the group-like elements of $kCN_1$ are exactly given by $N$

Sketch of Proof:

\[ \left\{ (e_i)_{i \in I}, (f_j)_{j \in J} \right\} \text{ bases for } (W, V) \Rightarrow \left\{ (e_i \otimes f_j)_{(i,j) \in I \times J} \right\} \text{ basis of } W \otimes V \]

b) => clear by definition of $m^*$

\[ m^* \left( \sum_{i=1}^{n} a_i n_i \right) = \sum_{i=1}^{n} a_i m^*(n_i) = \sum_{i=1}^{n} a_i \sigma_i \otimes m_i \otimes w_i \]

Definition B.3. Diagonalizable Groups
An algebraic group $G$ is called diagonalizable if

\[ G \cong \text{Spec}(kCN_1) = \text{D}(N) \]

for some finitely generated abelian group $N$

Lemma B.4.

\[ \text{Spec}(kCN_1) \text{ is given by } \mathbb{R} \rightarrow \text{Hom}_{\mathbb{R}}(N, \mathbb{R}^\times) \]

b) $D(CN_1)$ decomposes as

\[ D(CN_1) = \prod_{m_1} \mathbb{G}_m \times \mathbb{G}_m \times \mathbb{G}_m \times \cdots \times \mathbb{G}_m \]

c) $\chi(D(CN_1)) = N$

d) $G = \text{Spec} A$ diag $\iff A$ spanned by group-like elements

Sketch of proof:

\[ \mathbb{G}_m = \bigoplus \mathbb{Z} \oplus \bigoplus \mathbb{Z} \oplus \bigoplus \cdots \bigoplus \mathbb{Z} \bigoplus \mathbb{Z} \]

\[ \Rightarrow k[C] = k[C] \otimes \cdots \bigoplus \Rightarrow \text{ b) } \]

\[ \text{b) by B.2 a) and A.3. } \text{Spec}(A \otimes \mathbb{R}) = \text{Spec}(A) \times \text{Spec}(\mathbb{R}) \]

Theorem: B.5. Classification of diagonalizable groups

The contravariant functor

\[ D: \text{fg. abelian groups} \Rightarrow \text{diagonalizable groups} \]

\[ \text{is an equivalence of categories.} \]

(with inverse $G \mapsto \chi(G)$)
Sketch of proof:
1) essentially surj. by definition
2) fully faithful, i.e. \( \text{Hom}(N,N') \to \text{Hom}(\alpha N, \alpha N') \)
   
   \[ B.4 \text{ b) } \Rightarrow \text{reduce to } N, N' \in \mathbb{Z}^n \cup \{ \mathbb{Z} \text{ mod } \text{integers} \} \]
   
   (cyclic groups)
   
   \[ \Rightarrow \text{distinguish 4 possible cases} \]
   
   If \( N = N' = \mathbb{Z} \):
   
   \( \text{Hom}(N,N') = \text{Hom}(\mathbb{Z}, \mathbb{Z}) \cong \mathbb{Z} \text{ and} \)

   \[ \text{Hom}(\Gamma_{\mathbb{Z}}, \Gamma_{\mathbb{Z}}) = \text{Hom}(\mathbb{Z}^\times, \mathbb{Z}^\times) \cong \mathbb{Z} \text{ for all morphisms of Hopf algebras} \]

**Main Theorem B.6.** The following conditions are equivalent
a) \( G \) diagonalizable
b) every finite dimensional representation is a direct sum of characters
c) for every finite dimensional representation \( (V, \iota_r) \) of \( G \),
   
   \[ V = \bigoplus_{\iota \in \text{ex}(G)} V^\iota \]

For following talks, we will need:

**Corollary B.7.** \( G \) diag., \( V \) fin. \( k \)-vector space
Then:

\[ V = \bigoplus_{\iota \in \text{ex}(G)} V^\iota \text{ with } \bigoplus_{\iota \in \text{ex}(G)} \text{subspaces } V^\iota \]

\( \Rightarrow \text{representation of } G \) on \( V \)

\[ \text{"x(A)" - gradation of } V \]

Proof: Follows by (b) \( \iff \) (a) in B.6.

Indeed: \( G \) diagn \( \Rightarrow \) every \( (V, \iota_r) : r = \bigoplus_{\iota \in \text{ex}(G)} \) for some \( \iota \in \text{ex}(G) \)
"Define $V_m = V_{x_m}$ as given by $a$)
\[ \dim(V_m) \]
\[ \rightarrow \dim V = \sum_{m \in \text{ex}(a)} \dim (V_m) \]
\[ \text{Define } r = \bigoplus_{m \in \text{ex}(a)} \]
\[ \text{(corresponding to basis of eigenvectors given by $V_m$)} \]

**Sketch of proof B.6:**

We only show $a) \Rightarrow b)$ and $b) \Rightarrow c)$

"$a) \Rightarrow b)$"

Get diagonal and $\{V_m\}$ representation.

Aim: There is basis of $V$ s.t. $r(G) \in \text{Dn}(k)$

i.e. $r(G)(v) \in \text{Dn}(k) \neq R$

"all $r(G)$ are simultaneously diagonalizable"

Recall:

\[ r(G)(v) \in \text{Dn}(k) \Rightarrow r(G) \in \text{Dn} \]

$x(G) \in \text{Dn}$

In $\text{char}(k) = 0$ it suffices: all $r(G) \in r(G(k))$ simultaneously diag.

Fact from LA: $C$ commutative set of matrices $\Rightarrow$

(i.e. pairwise commutative):

separately diag. $\Rightarrow$ simultaneous diag.

**Steps of the proof:** (for $\text{char}(k) = 0$)

1. Every $g \in G(k)$ is semisimple (= diagonalizable)

   \[ G(k) = D(C) \cong G_m \times \prod_{i=1}^{r} \mathbb{M}_{m_i} \cong D_{n(k)} \]

   \[ \mathbb{M}_{m_i} \cong G_m \text{ by } \mathbb{Z} \rightarrow \mathbb{Z}[m] \]

2. Every $r(G) \in \text{GL}(k)$ is diagonalizable

   By Jordan decompos.: $r(G) = \rho r(G_s) = r(G_s)$

   \[ g = g_s \]
(3) \( r(G(k)) \) is a commutative set (i.e., elements commute pairwise).

\[ \forall g, h \in G(k), r(g) \cdot r(h) = r(hg) = r(hg) \cdot r(g) \]

\[ \text{Hom}_\text{cts}(N, k^*) \]

Fact (k)

\[ \Rightarrow \quad \forall (g) \in r(G(k)) \text{ simultaneously diagonal}, \quad \text{i.e. } \exists \mathbf{S} \in \text{GL}(k) : \mathbf{S}^{-1} (g) \mathbf{S} = \text{Diag}(k) \]

\[ (\text{char}(k) = 0) \]

\[ \Rightarrow \quad \text{(b) as } r \rightarrow S \circ r \circ S \text{ is isomorphism} \]

\[ \Rightarrow \quad \text{(c)} \]

As in LA, one can show that eigenvectors corresponding to different characters are linear independent \( \Rightarrow \) sum of eigenspaces is direct.