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Groups of multiplicative type

- base field k algebraically closed
- all algebraic groups affine: $G = \operatorname{Spec} A$

Definition

An algebraic group is of multiplicative type if it becomes diagonalizable over the seperable closure of the base field k. (i.e. $G_{k^{sep}} = G \otimes_k k^{sep}$ diag.)

$\overset{k \text{ alg. closed}}{\Longrightarrow}$ multiplicative type = diagonalizable

Hence:

Diagonalizable Groups

V always finite dimensional k-vector space

(A) Characters & diagonalizable representations

Recall A.1.

A morphism $r: G \longrightarrow GL_V \cong GL_n$ is called a representation.

Definition A.2.

A 1-dim. representation (i.e. $\dim V = 1$)

$$\chi : G \longrightarrow GL_1 \cong \mathbb{G}_m$$

is called a character of G.

Remark A.3. Group-like elements

$$\{\chi : G \longrightarrow \mathbb{G}_m\} \longleftrightarrow \tilde{\chi} : k[T, T^{-1}] \longrightarrow A$$

(k-scheme morphism resp. m, e, i) (k-alg morph. resp. m^*, e^*, i^*)

i.e. given by $T \longmapsto a$

s.t.

obtain commut. group structure by

$\chi + \chi' : G(R) \longrightarrow R^\times$,

$g \longmapsto \chi(g)\,\chi'(g)$

$\xleftrightarrow{1:1}$ $\{ a \in A^\times$ with $m^*(a) = a \otimes a \}$ ($m^*(T) = T \otimes T$)

"group-like elements"

$X(G) :=$ abelian group of χ on G

Example A.4. Diagonal representations

Let $V = k^n$

1) $\chi : G \longrightarrow \mathbb{G}_m$ defines representation r of G on V by

$$r : G \longrightarrow GL_V \cong GL_n$$

$$G(R) \ni g \longmapsto \begin{pmatrix} \chi(g) & & \\ & \ddots & \\ & & \chi(g) \end{pmatrix} \in D_n(R) \subseteq GL_n(R), \quad v \in V)$$

i.e. $r(g) \cdot v = \chi(g) v \quad \forall g \in G(R), v \in V$

"G acts on V through χ"

2) More generally: $\chi_1, \dots, \chi_n$ ($\chi_i = \chi_j$ allowed)

$$\bigoplus_{i=1}^{n} \chi_i =: r : G \longrightarrow GL_n$$

$$G(R) \ni g \longmapsto \begin{pmatrix} \chi_1(g) & & \\ & \ddots & \\ & & \chi_n(g) \end{pmatrix} \in D_n(R) \subseteq GL_n(R)$$

$\in R^\times$

Definition A.5: eigenspace

$(V, r) : G \longrightarrow GL_V$, $\chi \in X(G)$

If there exists $v \in V$ s.t. $\forall g \in G(R)\ \forall R$

$$r(g) \cdot v_R = \chi(g) \cdot v_R \quad (v_R \in V \otimes R)$$

then "G acts on $\langle v \rangle \subset V$ through χ"

$$V_\chi = \{ v \in V \mid r(g) v_R = \chi(g) \cdot v_R \ \forall g \in G(R) \ \forall R \}$$

greatest subspace of V on which G acts through χ

"eigenspace for G with character χ"

Example: $r = \bigoplus_{i=1}^{n} \chi_i$ (as in A.4.2)) $\Rightarrow e_i \in k^n$ is eigenvector of r corresponding to χ_i $\Rightarrow V = \bigoplus_{\chi \in X(G)} V_\chi$ (almost all V_χ zero)

Question: Let $(V, r): G \longrightarrow GL_V$ be arbitrary
Under which assumption there exists a basis for V
s.t. $r(G) \subseteq D_n \cong \mathbb{G}_m^n$
$\iff r = \bigoplus_{i=1}^{n} \chi_i$ for some $\chi_i \in X(G)$

Answer: G diagonalizble group
In fact:
Theorem: Equivalent:
a) G diagonalizble
b) every finite dimensional representation is direct sum of characters
c) for every representation (V, r) of G: $V = \bigoplus_{\chi \in X(G)} V_\chi$

(B) Diagonalizble Groups

Let N be finitely generated abelian group.
Then $k[N] :=$ k-vector space with basis N
(i.e. $k[N] \ni x = \sum_{i=1}^{n} a_i n_i$ $\underline{a_i \in k}$, $\underline{n_i \in N}$, $n \in \mathbb{N}$)

1) is finitely generated k-algebra by extending multiplication of N
2) is Hopf algebra by setting
$m^*(n) = n \otimes n$, $e^*(n) = 1_N$, $i^*(n) = n^{-1}$

Example B.1.
1) $N = \mathbb{Z}$: $k[\mathbb{Z}] \ni x = \sum_{i \in \mathbb{Z}} a_i \cdot i$, $a_i \in k$, $a_i = 0$ f.a.a. $i \in \mathbb{Z}$
$\Rightarrow k[\mathbb{Z}] \cong k[T, T^{-1}]$ as Hopf algebras

2) $N = \mathbb{Z}/n\mathbb{Z}$: $k[\mathbb{Z}/n\mathbb{Z}] \ni x = \sum_{i=1}^{n-1} a_i \cdot i \Rightarrow k[\mathbb{Z}/n\mathbb{Z}] \cong k[T]/(T^n - 1)$

Lemma B.2.

a) N_1, N_2 f.g. ab. gr. : $k[N_1 \times N_2] \cong k[N_1] \otimes k[N_2]$ as Hopf algebras

b) The group-like elements of $k[N]$ are exactly given by N

<u>Sketch of Proof:</u>

a) $(e_i)_{i\in I}$, $(f_j)_{j\in J}$ bases for W, V (k-vector spaces) $\Rightarrow (e_i \otimes f_j)_{(i,j)\in I\times J}$ basis of $W \otimes V$

b) $\overset{k}{\Leftarrow}$ clear by definition of m^*

$\overset{k}{\Rightarrow}$ $m^*\left(\sum_{i=1}^{n} a_i n_i\right) \overset{\text{def}}{=} \sum_{i=1}^{n} a_i m^*(n_i) = \sum_{i=1}^{n} a_i (m_i \otimes m_i) \overset{!}{=} \sum_{i,j}^{n} a_i a_j (m_i \otimes m_j)$

$a_i^2 = 0,\ a_i a_j = 0$

Definition B.3 Diagonalizable Groups

An algebraic group G is called diagonalizable if

$$G \cong \operatorname{Spec}(k[N]) =: D(N)$$

for some finitely generated abelian group N

Lemma B.4.

a) $D(N)$ is given by $\underset{k\text{-alg}}{R} \longmapsto \underset{\text{Grps}}{\operatorname{Hom}_{\text{Grps}}(N, R^\times)}$

b) $D(N)$ decomposes as

$$D(N) \cong \mathbb{G}_m \times \dots \times \mathbb{G}_m \times \mu_{n_1} \times \dots \times \mu_{n_r}$$

c) $X(D(N)) = N$

d) $G = \operatorname{Spec} A$ diag $\Leftrightarrow$ A spanned by group-like elements

<u>Sketch of proof:</u>

b) $M = \mathbb{Z} \oplus \dots \oplus \mathbb{Z} \oplus \mathbb{Z}/n_1\mathbb{Z} \oplus \dots \oplus \mathbb{Z}/n_r\mathbb{Z}$

$\Rightarrow k[M] \cong k[\mathbb{Z}] \otimes \dots\dots\dots \Rightarrow$ b)

$\uparrow$ B.2 a) $\qquad \uparrow \operatorname{Spec}(A \otimes B) = \operatorname{Spec}(A) \times \operatorname{Spec}(B)$ & B.1.

c) By B.2 b) and A.3.

Theorem: B.5 Classification of diagonalizable groups

The contravariant functor

$$D: \{\text{f.g. abelian groups}\}^{op} \longrightarrow \{\text{diagonalizable groups}\}$$
$$M \longmapsto D(M)$$

is an equivalence of categories.

(with inverse $G \longmapsto X(G)$)

Sketch of proof:

1) essentially surj. by definition

2) fully faithful, ie. $\mathrm{Hom}(N,N') \xrightarrow{\sim} \mathrm{Hom}(D(N), D(N'))$

B.4 b) $\Rightarrow$ reduce to $N, N' \in \{\mathbb{Z}\} \cup \{\mathbb{Z}/n\mathbb{Z} \mid n \in \mathbb{N}\}$ (cyclic groups)

$\Rightarrow$ distinguish 4 possible cases

If $N = N' = \mathbb{Z}$:

$\mathrm{Hom}(N,N') = \mathrm{Hom}(\mathbb{Z},\mathbb{Z}) \cong \mathbb{Z}$ and

$\underset{\substack{\| \\ D(\mathbb{Z})}}{\mathrm{Hom}(\mathbb{G}_m, \mathbb{G}_m)} = \underset{\substack{\nearrow \\ \text{morphisms} \\ \text{of Hopf algebras}}}{\mathrm{Hom}}(k[T,T^{-1}], k[T,T^{-1}]) \cong \{X^i \mid i \in \mathbb{Z}\} \cong \mathbb{Z}$

Main Theorem B.6. The following conditions are equivalent

a) G diagonalizable

b) every finite dimensional representation is a direct sum of characters

c) for every finite dimensional representation (V,r) of G,

$$V = \bigoplus_{\chi \in X(G)} V_\chi$$

For following talks, we will need:

Corollary B.7.: G diag., V f.d k-vector space

Then:

$V = \bigoplus_{m \in X(G)} V_m$ with $\{V_m\}_{m \in X(G)}$ subspaces $\longleftrightarrow$ representation of G on V

" $X(G)$-gradation of V"

Proof: Follows by (b)$\Leftrightarrow$(d) in B.6.

Indeed: G diagn $\overset{a)}{\Longrightarrow}$ every (V,r): $r = \bigoplus_{i=1}^{n} \chi_i$ for some $\chi_i \in X(G)$

"$\leftarrow$" Define $V_m = V_\chi$ as given by d)

"$\rightarrow$" $\dim V = \sum_{m \in X(G)} \dim(V_m)$. Define $r := \bigoplus_{m \in X(G)} \bigoplus_{i=1}^{\dim(V_m)} m$

(corresponding to basis of eigenvectors given by V_m)

<u>Sketch of proof B.6:</u>

We only show a) $\Rightarrow$ b) and b) $\Rightarrow$ c)

"a) $\Rightarrow$ b)"

G diagn. and (V, r) representation.

Aim: There is basis of V s.t. $r(G) \subseteq \mathbb{D}_n$

i.e. $r(G(R)) \subseteq \mathbb{D}_n(R)$ $\forall R$

"all $r(g)$ are simultaneously diagonalizable"

Recall:

(If $\mathrm{char}(k) = 0$) $r(G(k)) \subset \mathbb{D}_n(k) \Rightarrow r(G) \subseteq \mathbb{D}_n$

⚠ $\Rightarrow G$ smooth

$X(G) \subseteq X$ dense

In $\mathrm{char}(k) = 0$ it suffices: all $r(g) \in r(G(k))$ simultaneously diag.

Fact from LA: C commutative set of matrices (*) (i.e. pairwise commutative):

seperatedly diagn. $\Leftrightarrow$ simult. diagn.

Steps of the proof: (For $\mathrm{char}(k) = 0$)

① Every $g \in G(k)$ is semisimple (= diagonalizable)

$G \cong D(M) \cong \mathbb{G}_m^d \times \prod_{i=1}^{r} \mu_{n_i} \subseteq \mathbb{G}_m^{d+r} \cong \mathbb{D}_{d+r}$

$\mu_{n_i} \subseteq \mathbb{G}_m$ by $\mathbb{Z} \to \mathbb{Z}/n_i\mathbb{Z}$

② Every $r(g) \in GL_n(k)$ is diagonalizable

By Jordan decompos.: $r(g) = r(g_s) = r(g)_s$

$g = g_s$

③ $r(G(k))$ is a commutative set (i.e. elements commute pairwise)

$g,h \in G(k)$ is abelian $\Rightarrow r(g)r(h) = r(gh) = r(hg) = r(h)\cdot r(g)$

$\quad\quad \|$

$\mathrm{Hom}_{grps}(N, k^{\times})$

Fact $(*)$

$\Longrightarrow$ All $r(g) \in r(G(k))$ simultaneously diagon.

i.e. $\exists\, S \in Gl_n(k) : S^{-1} r(g) S \in D_n(k)$

$\forall g \in G(k)$

$(\mathrm{char}(k) = 0)$

$\Longrightarrow$ (b) as $r \longmapsto S^{-1} \circ r \circ S$ is isomorphism

" (b) $\Rightarrow$ (c) "

As in LA, one can show that eigenvectors corresponding to different characters are linear independent $\Rightarrow$ sum of eigenspaces is direct